PROJECTION OPERATOR METHOD FOR THE SU(2) CLEBSCH-GORDAN COEFFICIENTS AND MAP OF EVEN SPIN-TENSORS INTO TENSORS AND FUNCTIONS ON EUCLIDEAN 3-SPACE IN THE THEORY OF SU(2)

BY

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DEDICATION

This dissertation is dedicated to the author's mother, who died on November 11, 1975. Knowing that she had cancer and might die very soon, she still encouraged her son to go abroad to the University of Florida. This dissertation is also dedicated to the author's father Mr. Geng Hwei Hwang and to the author's wife Mrs. Yeh-han Hwang.

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PROJECTION OPERATOR METHOD FOR THE SU(2) CLEBSCH-GORDAN COEFFICIENTS AND MAP OF EVEN SPIN-TENSORS INTO TENSORS AND FUNCTIONS ON EUCLIDEAN 3-SPACE IN THE THEORY OF SU(2)

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Clebsch-Gordan coefficients have been used in quantum physics to accomplish the addition of angular momenta. More recently, the Clebsch-Gordan coefficients for SU(3), SU(6) and other groups have been used in the theory of elementary particles. In this presentation, we emphasize more the mathematical methods related to the derivation of the Clebsch-Gordan coefficients, rather than the coefficients themselves. A direct procedure is given for obtaining the canonical basis of the tensor product of two irreducible representations of SU(2) without using these coefficients. The Clebsch-Gordan coefficients are then derived from these results only as an afterthought.

The first chapter is devoted to giving a complete discussion, mainly quoted from C.P. Luehr's notes (not published), of how to derive the Clebsch-Gordan coefficients. Then the normalized canonical basis for the irreducible representations contained in the tensor product of two irreducible representations of SU(2) are found. This is accomplished by a direct construction in spin-tensor spaces (tensor spaces associated with the two dimensional unitary space). The method is completely

algebraic, involving only tensor operations and calculations with projection operators contained in the group algebras of the permutation groups. Along with this, one of the main new results, called the Inner Product Theorem is proved. This theorem relates the inner products in a certain tensor space with the inner product in the image space under the principal map used in the construction of the canonical basis.

In the second chapter, the spin-tensor formulation of the representation theory of SU(2) is related to two other formulations by investigating maps which relate the algebra of symmetric even spin-tensors to the algebra of symmetric traceless tensors on Euclidean 3-space and the analysis of homogeneous harmonic functions on Euclidean 3-space. Projection operators onto the subspaces of symmetric spin tensors are mapped into projection operators onto subspaces of symmetric traceless tensors. A theorem is proved which relates the inner products of elements in the domains with the corresponding inner products of their images under these maps. Starting with suitable bases in the spin-tensor spaces, these maps produce bases in the tensor and function spaces. The last mentioned bases are the spherical harmonics.

It is hoped that the present approach over the three different spaces, though far from being comprehensive, may provide an additional basis for such comparative studies.

CHAPTER 1

PROJECTION OPERATOR METHOD FOR THE SU(2) CLEBSCH-GORDAN COEFFICIENTS

The Unitary Space U_2

The space U_2 is a two dimensional unitary space, i.e., it is a two dimensional vector space over the field C of complex numbers with an inner product $\langle u|v\rangle$, which is a complex number associated with every pair u,v in U_2 such that

(1) $\overline{\langle u|v\rangle} = \langle v|u\rangle$

(2) $\langle u|\alpha v+\beta w\rangle = \alpha\langle u|v\rangle+\beta\langle u|w\rangle$

$\langle \alpha u+\beta v|w\rangle = \bar{\alpha}\langle u|w\rangle+\bar{\beta}\langle v|w\rangle$

(3) $\langle u|u\rangle > 0$ if $u \neq 0$, and

$\langle u|u\rangle = 0$ if $u = 0$ (non-negativity)

for all u,v,w in U_2 and α,β in C, where $\bar{\alpha}$ is the complex conjugate of α and 0 is the zero vector of U_2.

The Conjugate Space $\bar{U}_2$ of U_2

The conjugate space $\bar{U}_2$ associated with U_2 is the dual space to U_2 . i.e., the space of linear functionals on U_2, provided with a conjugation operation, i.e., a map; u in $U_2 \rightarrow \bar{u}$ in $\bar{U}_2$, uniquely defined by the equation $\bar{u} \circ w = \langle u|w\rangle$ for all w in U_2 where $\bar{u} \circ w \equiv w \circ \bar{u}$ denotes the value resulting from the action of the linear functional $\bar{u}$ on the vector w. The inverse map, $\bar{u}$ in $U_2 \rightarrow \bar{\bar{u}}=u$ in U_2, is also called a conjugation operation.

Some Important Properties

(1) The conjugate space $\overline{U}_2$ is a two dimensional vector space over C.

(2) The conjugation map, u in $U_2 \rightarrow \overline{u}$ in $\overline{U}_2$, is one-to-one from U_2 onto $\overline{U}_2$ and similarly for the inverse map.

(3) $\overline{\overline{u}} = u$

(4) $\overline{(\alpha u+\beta v)} = \overline{\alpha}\ \overline{u} + \overline{\beta}\ \overline{v}$

(5) $\overline{(\alpha\overline{u}+\beta\overline{v})} = \overline{\alpha}\ \overline{\overline{u}} + \overline{\beta}\ \overline{\overline{v}} = \overline{\alpha}\ u + \overline{\beta}\ v$

(6) $\overline{(\overline{u} \circ v)} = u \circ \overline{v}$

for all u,v in U_2 and α,β in C. Note that properties (4) and (5) state the antilinearity of the conjugation operation.

The r-th Order Tensor Spaces $U_2^{\otimes r}$ and $\overline{U}_2^{\otimes r}$

An r-th order tensor over U_2 is any complex valued r-linear functional $T(\overline{w}_1,\overline{w}_2,\cdots,\overline{w}_r)$ of the arguments $\overline{w}_1,\overline{w}_2,\cdots,\overline{w}_r$ in $\overline{U}_2$. The notation T will be used for the tensor, and the value which results from the action of T on $\overline{w}_1,\overline{w}_2,\cdots,\overline{w}_r$ will be denoted by $(T|\overline{w}_1,\overline{w}_2,\cdots,\overline{w}_r) \equiv T(\overline{w}_1,\overline{w}_2,\cdots,\overline{w}_r)$.

The space $U_2^{\otimes r}$ is the set of all r-th order tensors over U_2. The sum S+T of S and T in $U_2^{\otimes r}$ is an element of $U_2^{\otimes r}$ uniquely defined by the equation

$$(S+T|\overline{w}_1,\overline{w}_2,\cdots,\overline{w}_r) = (S|\overline{w}_1,\overline{w}_2,\cdots,\overline{w}_r) + (T|\overline{w}_1,\overline{w}_2,\cdots,\overline{w}_r)$$

for all values of the arguments $\overline{w}_1,\overline{w}_2,\cdots,\overline{w}_r$ in $\overline{U}_2$. For any α in C, the product αT of α with any T in $U_2^{\otimes r}$ is another element of $U_2^{\otimes r}$ uniquely defined by the equation

$$(\alpha T|\overline{w}_1,\overline{w}_2,\cdots,\overline{w}_r) = \alpha(T|\overline{w}_1,\overline{w}_2,\cdots,\overline{w}_r)$$

for all values of $\overline{w}_1,\overline{w}_2,\cdots,\overline{w}_r$ in $\overline{U}_2$. The zero tensor 0 in $U_2^{\otimes r}$ is a unique element defined by the equation

$$(0|\overline{w}_1,\overline{w}_2,\cdots,\overline{w}_r) = 0$$

for all values of $\overline{w}_1,\overline{w}_2,\cdots,\overline{w}_r$ in $\overline{U}_2$. It can be verified that $U_2^{\otimes r}$ is a vector space over C with the above defined operations of addition of tensors and multiplication of tensors by complex numbers.

In the special case of first order tensors, it can be shown that there is a one-to-one correspondence between elements T in $U_2^{\otimes 1}$ and elements t in U_2 such that $(T|\overline{w}_1) = t \circ \overline{w}_1$. It follows that $U_2^{\otimes 1}$ and U_2 are isomorphic as vector spaces, and we shall simply treat them as identical; thus first order tensors over U_2 are vectors in U_2 itself. Likewise the space of zero order tensors over U_2 is C.

Given a set of vectors $u_1,u_2,\cdots,u_r$ in U_2, their tensor product, written in the following alternate notations

$$\bigotimes_{k=1}^{r} u_k \equiv u_1 \otimes u_2 \otimes \cdots \otimes u_r \equiv u_1u_2 \cdots u_r$$

is an element of $U_2^{\otimes r}$ defined by the equation

$$(u_1u_2 \cdots u_r | \overline{w}_1,\overline{w}_2,\cdots,\overline{w}_r) = u_1\circ\overline{w}_1 u_2\circ\overline{w}_2 \cdots u_r\circ\overline{w}_r$$

for all values of $\overline{w}_1,\overline{w}_2,\cdots,\overline{w}_r$ in $\overline{U}_2$. In the special case of a tensor product of u with itself r times we can also use the notation

$$u^{\otimes r} \equiv \bigotimes_{k=1}^{r} u \equiv u\, u \cdots u$$

Let g_1, g_2 be an orthonormal basis for U_2, thus they satisfy

$$\overline{g}_a \circ g_b = \delta_{ab}$$

for a=1,2 and b=1,2 where δ_{ab} is the Kronecker delta.

Theorem 1: $g_{a_1} g_{a_2} \cdots g_{a_r}$ for all possible values $a_1=1,2; a_2=1,2; \cdots; a_r=1,2$ is a basis for $U_2^{\otimes r}$, hence $U_2^{\otimes r}$ has the dimension 2^r.

Proof: First, we claim that $g_{a_1} g_{a_2} \cdots g_{a_r}$ for all possible values $a_1=1,2; a_2=1,2; \cdots ; a_r=1,2$ are linearly independent. Suppose that

$$\sum_{a_1,a_2,\cdots,a_r} \alpha_{a_1 a_2 \cdots a_r} g_{a_1} g_{a_2} \cdots g_{a_r} = 0$$

Then for any set of values for $b_1=1,2; b_2=1,2; \cdots ; b_r=1,2$.

$$\begin{aligned} 0 &= \Big(\sum_{a_1,a_2,\cdots,a_r} \alpha_{a_1 a_2 \cdots a_r} g_{a_1} g_{a_2} \cdots g_{a_r} \Big| \overline{g}_{b_1}, \overline{g}_{b_2}, \cdots, \overline{g}_{b_r} \Big) \\ &= \sum_{a_1,a_2,\cdots,a_r} \alpha_{a_1 a_2 \cdots a_r} (g_{a_1} g_{a_2} \cdots g_{a_r} | \overline{g}_{b_1}, \overline{g}_{b_2}, \cdots, \overline{g}_{b_r}) \\ &= \sum_{a_1,a_2,\cdots,a_r} \alpha_{a_1 a_2 \cdots a_r} \delta_{a_1 b_1} \delta_{a_2 b_2} \cdots \delta_{a_r b_r} \\ &= \alpha_{b_1 b_2 \cdots b_r} \end{aligned}$$

Therefore they are linearly independent.

Next we claim that every element in $U_2^{\otimes r}$ can be expressed uniquely by $g_{a_1} g_{a_2} \cdots g_{a_r}$, for all $a_1=1,2; a_2=1,2; \cdots ; a_r=1,2$. Let the arguments $\overline{w}_i = \sum_{a_i=1}^{2} (\overline{g}_{a_i} g_{a_i} \cdot \overline{w}_i)$ for $i=1,2,\cdots,r$ and T be an arbitrary tensor in $U_2^{\otimes r}$. Then

$$\begin{aligned}
&(T|\bar{w}_1,\bar{w}_2,\cdots,\bar{w}_r) \\
&= (T|\sum_{a_1=1}^{2}\bar{g}_{a_1}g_{a_1}\cdot\bar{w}_1,\ \sum_{a_2=1}^{2}\bar{g}_{a_2}g_{a_2}\cdot\bar{w}_2,\cdots,\ \sum_{a_r=1}^{2}\bar{g}_{a_r}g_{a_r}\cdot\bar{w}_r) \\
&= \sum_{a_1,a_2,\cdots,a_r}(T|\bar{g}_{a_1},\bar{g}_{a_2},\cdots,\bar{g}_{a_r})g_{a_1}\cdot\bar{w}_1\ g_{a_2}\cdot\bar{w}_2\cdots g_{a_r}\cdot\bar{w}_r \\
&= \sum_{a_1,a_2,\cdots,a_r}T_{a_1a_2\cdots a_r}(g_{a_1}g_{a_2}\cdots g_{a_r}|\ \bar{w}_1,\bar{w}_2,\cdots,\bar{w}_r) \\
&= (\sum_{a_1,a_2,\cdots,a_r}T_{a_1a_2\cdots a_r}\ g_{a_1}g_{a_2}\cdots g_{a_r}|\bar{w}_1,\bar{w}_2,\cdots,\bar{w}_r)
\end{aligned}$$

for all values of $\bar{w}_1,\bar{w}_2,\cdots,\bar{w}_r$, where $T_{a_1a_2\cdots a_r} = (T|\bar{g}_{a_1},\bar{g}_{a_2},\cdots,\bar{g}_{a_r})$. Therefore

$$T = \sum_{a_1,a_2,\cdots,a_r}T_{a_1a_2\cdots a_r}\ g_{a_1}g_{a_2}\cdots g_{a_r} \qquad \text{Q.E.D.}$$

Similarly to tensors over U_2, tensors over $\bar{U}_2$ can be defined. An r-th order tensor over $\bar{U}_2$ is any complex valued r-linear functional $T'(w_1,w_2,\cdots,w_r)$ of the arguments $w_1,w_2,\cdots,w_r$ in U_2. The notation $(T'|w_1,w_2,\cdots,w_r) \equiv T'(w_1,w_2,\cdots,w_r)$ will be used. The space $\bar{U}_2^{\otimes r}$ is the set of all r-th order tensors over $\bar{U}_2$. The sum $S'+T'$ of two tensors in $\bar{U}_2^{\otimes r}$, and the product $\alpha T'$ of any α in C with T' in $\bar{U}_2^{\otimes r}$, are defined analogously to the corresponding operations in $U_2^{\otimes r}$, furthermore the zero tensor $0'$ in $\bar{U}_2^{\otimes r}$ is defined in a similar way as 0 in $U_2^{\otimes r}$. It follows that $\bar{U}_2^{\otimes r}$ is a vector space over C.

First order tensors over $\bar{U}_2$ will be identified with vectors in $\bar{U}_2$, and the space of zero order tensors over $\bar{U}_2$ will be C.

A tensor product of the form

$$\overset{r}{\underset{k=1}{\otimes}} \bar{u}_k \equiv \bar{u}_1 \otimes \bar{u}_2 \otimes \cdots \otimes \bar{u}_r \equiv \bar{u}_1\bar{u}_2 \cdots \bar{u}_r$$

with $\bar{u}_1,\bar{u}_2,\cdots,\bar{u}_r$ in $\bar{U}_2$ is a tensor in $\bar{U}_2^{\otimes r}$ defined by the equation

$$(\bar{u}_1\bar{u}_2\cdots\bar{u}_r \mid w_1,w_2,\cdots,w_r) = \bar{u}_1\circ w_1\bar{u}_2\circ w_2\cdots\bar{u}_r\circ w_r$$

for all values of $w_1,w_2,\cdots,w_r$ in U_2. Taking a basis $\bar{g}_1,\bar{g}_2$ of $\bar{U}_2$, it follows that $\bar{g}_a\bar{g}_{a_2}\cdots\bar{g}_{a_r}$ is a basis for $\bar{U}_2^{\otimes r}$. Thus $\bar{U}_2^{\otimes r}$ also has the dimension 2^r.

For any T in $U_2^{\otimes r}$, its conjugate $\bar{T}$ is an element of $\bar{U}_2^{\otimes r}$ defined by the equation

$$(\bar{T} \mid w_1,w_2,\cdots,w_r) = \overline{(T \mid \bar{w}_1,\bar{w}_2,\cdots,\bar{w}_r)}$$

for all values of $w_1,w_2,\cdots,w_r$ in U_2. Similarly for any S' in $\bar{U}_2^{\otimes r}$, its conjugate is an element of $U_2^{\otimes r}$ defined by the equation

$$(\bar{S}' \mid \bar{w}_1,\bar{w}_2,\cdots,\bar{w}_r) = \overline{(S' \mid w_1,w_2,\cdots,w_r)}$$

for all values of $\bar{w}_1,\bar{w}_2,\cdots,\bar{w}_r$ in $\bar{U}_2$.

Some Important Properties of the Conjugation Operation in $U_2^{\otimes r}$ and $\bar{U}_2^{\otimes r}$

(1) The conjugation map; T in $U_2^{\otimes r} \rightarrow \bar{T}$ in $\bar{U}_2^{\otimes r}$, is one-to-one from $U_2^{\otimes r}$ onto $\bar{U}_2^{\otimes r}$, and vice versa, hence $\bar{\bar{T}} = T$.

(2) $\overline{(\alpha S+\beta T)} = \bar{\alpha}\,\bar{S} + \bar{\beta}\,\bar{T}$

(3) $\overline{(\alpha\overline{S}+\beta\overline{T})} = \overline{\alpha}\ \overline{\overline{S}} + \overline{\beta}\ \overline{\overline{T}} = \overline{\alpha}\ S + \overline{\beta}\ T$

(4) $\overline{(u_1u_2\cdots u_r)} = \overline{u}_1\overline{u}_2\cdots\overline{u}_r$

(5) $\overline{(\overline{u}_1\overline{u}_2\cdots\overline{u}_r)} = \overline{\overline{u}}_1\overline{\overline{u}}_2\cdots\overline{\overline{u}}_r = u_1u_2\cdots u_r$

(6) $0' = \overline{0}$, and $0 = \overline{0}'$.

From now on we shall express elements in $\overline{U}_2^{\otimes r}$ as the conjugate of elements in $U_2^{\otimes r}$, i.e., instead of T' we shall write $\overline{T}$, with T in $U_2^{\otimes r}$.

Tensor Products

The tensor product $S \otimes T \equiv ST$ of S in $U_2^{\otimes r}$ and T in $U_2^{\otimes s}$ is an element of $U_2^{\otimes(r+s)}$ defined by the equation

$$(ST\,|\,\overline{w}_1,\overline{w}_2,\cdots,\overline{w}_r,\overline{w}_{r+1},\overline{w}_{r+2},\cdots,\overline{w}_{r+s})$$

$$= (S\,|\,\overline{w}_1,\overline{w}_2,\cdots,\overline{w}_r)\ (T\,|\,\overline{w}_{r+1},\overline{w}_{r+2},\cdots,\overline{w}_{r+s})$$

for all values of $\overline{w}_1,\overline{w}_2,\cdots,\overline{w}_r,\overline{w}_{r+1},\overline{w}_{r+2},\cdots,\overline{w}_{r+s}$ in $\overline{U}_2$. The tensor product $\overline{S} \otimes \overline{T}$ is defined analogously. The following properties can easily be verified for tensors R, S and T of arbitrary orders

(1) $(R \otimes S) \otimes T = R \otimes (S \otimes T)$

(2) $(\alpha R+\beta S) \otimes T = \alpha(R \otimes T) + \beta(S \otimes T)$

(3) $R \otimes (\alpha S+\beta T) = \alpha(R \otimes S) + \beta(R \otimes T)$

(4) $\overline{(S \otimes T)} = \overline{S} \otimes \overline{T}$

and also similar properties with R, S, T replaced by $\overline{R}$, $\overline{S}$, $\overline{T}$. In (2) it is assumed that R and S are in the same tensor space, and in (3) that S and T are in the same tensor space. The quantity in (1) is simply and unambiguously written $R \otimes S \otimes T = (R \otimes S) \otimes T = R \otimes (S \otimes T)$.

Inner Product in $U_2^{\otimes r}$

For $\bar{S}$ in $\bar{U}_2^{\otimes r}$ and T in $U_2^{\otimes r}$, their dot product $\bar{S} \odot T \equiv T \odot \bar{S}$ is defined in the special case where $\bar{S} = \bar{u}_1\bar{u}_2\cdots\bar{u}_r$ and $T = v_1v_2\cdots v_r$ as

$$(\bar{u}_1\bar{u}_2\cdots\bar{u}_r) \odot (v_1v_2\cdots v_r) = (\bar{u}_1 \circ v_1)(\bar{u}_2 \circ v_2)\cdots(\bar{u}_r \circ v_r)$$

and then uniquely extended to arbitrary $\bar{S}$ and T by the assumption of the properties:

(1) $(\alpha\bar{Q}+\beta\bar{S}) \odot T = \alpha\bar{Q} \odot T + \beta\bar{S} \odot T$

(2) $\bar{S} \odot (\alpha R+\beta T) = \alpha\bar{S} \odot R + \beta\bar{S} \odot T$

for R and T in $U_2^{\otimes r}$ and $\bar{Q}$ and $\bar{S}$ in $\bar{U}_2^{\otimes r}$.

The following properties can be proved:

(3) $\overline{(\bar{S} \odot T)} = \bar{T} \odot S$

(4) $\bar{T} \odot T > 0$ if $T \neq 0$, and $\bar{T} \odot T = 0$ if $T = 0$.

(5) $(\bar{S} \times \bar{Q}) \odot (T \times R) = (\bar{S} \odot T)(\bar{Q} \odot R)$

for S,T in $U_2^{\otimes r}$, and Q,R in $U_2^{\otimes s}$.

An inner product $\langle S|T \rangle$ for S,T in $U_2^{\otimes r}$ can be defined as

$$\langle S|T \rangle = \bar{S} \odot T$$

This inner product in $U_2^{\otimes r}$ has the same properties as the inner product in U_2 as previously listed.

The Group Ring CS_r Over the Complex Numbers C

Given T in $U_2^{\otimes r}$ and p in S_r, with S_r the permutation group on the set $H_r = \{1,2,\cdots,r\}$, pT is another tensor in $U_2^{\otimes r}$ defined as

$$(pT \mid \bar{w}_1,\bar{w}_2,\cdots,\bar{w}_r) = (T \mid \bar{w}_{p1},\bar{w}_{p2},\cdots,\bar{w}_{pr})$$

for all $\bar{w}_1,\bar{w}_2,\cdots,\bar{w}_r$ in $\bar{U}_2$. It follows that p operates on a tensor product $u_1u_2\cdots u_r$ as

$$p(u_1u_2\cdots u_r) = u_{\hat{p}1}u_{\hat{p}2}\cdots u_{\hat{p}r}$$

where $\hat{p}$ is the inverse permutation of p.

For any given set of complex numbers α_p for each p in S_r, the quantity $P = \sum_{p\varepsilon S_r} \alpha_p p$ is defined as an operator on tensors T in $U_2^{\otimes r}$ as follows

$$PT = (\sum_{p\varepsilon S_r} \alpha_p p)T = \sum_{p\varepsilon S_r} \alpha_p(pT)$$

If M is the subset of all p in S_r for which $\alpha_p \neq 0$ we can write $P = \sum_{p\varepsilon M} \alpha_p p$. In the set CS_r of all operators of the form $P = \sum_{p\varepsilon S_r} \alpha_p p$, we may define multiplication by scalars γ in C as

$$\gamma P = \gamma(\sum_{p\varepsilon S_r} \alpha_p p) = \sum_{p\varepsilon S_r} (\gamma\alpha_p)p \ ,$$

addition with other elements $Q = \sum_{p\varepsilon S_r} \beta_p p$ as

$$P+Q = (\sum_{p\varepsilon S_r} \alpha_p p) + (\sum_{p\varepsilon S_r} \beta_p p) = \sum_{p\varepsilon S_r} (\alpha_p+\beta_p)p$$

and multiplication of two elements of CS_r as

$$PQ = (\sum_{p\varepsilon S_r} \alpha_p p)(\sum_{q\varepsilon S_r} \beta_q q) = \sum_{p,q\varepsilon S_r} (\alpha_p\beta_q)pq$$

The set CS_r with these three operations is called the group ring of S_r.

We may define the complex conjugate $\overline{P}$ of any $P = \sum_{p\in S_r} \alpha_p p$ in CS_r as

$$\overline{P} = \sum_{p\in S_r} \overline{\alpha}_p p$$

Also define $\hat{P}$ as

$$\hat{P} = \sum_{p\in S_r} \alpha_p \hat{p}$$

Similarly for $\overline{T}$ in $U_2^{\otimes r}$ and $p\in S_r$, we define $p\overline{T}$ as

$$(p\overline{T} \,|\, w_1, w_2, \cdots, w_r) = (\overline{T} \,|\, w_{p1}, w_{p2}, \cdots, w_{pr})$$

for all $w_1, w_2, \cdots, w_r$ in U_2. Also $P = \sum_{p\in S_r} \alpha_p p$ operates on $\overline{T}$ as

$$P\overline{T} = \Big(\sum_{p\in S_r} \alpha_p p\Big)\overline{T} = \sum_{p\in S_r} \alpha_p (p\overline{T})$$

Theorem 1:

(1) $(\alpha P+\beta Q)T = \alpha(PT)+\beta(QT)$

(2) $(\alpha P+\beta Q)\overline{T} = \alpha(P\overline{T})+\beta(Q\overline{T})$

(3) $P(\alpha T+\beta S) = \alpha(PT)+\beta(PS)$

(4) $P(\alpha\overline{T}+\beta\overline{S}) = \alpha(P\overline{T})+\beta(P\overline{S})$

(5) $\overline{\overline{P}} = P$

(6) $\overline{PT} = \overline{P}\,\overline{T}$

(7) $\overline{P\overline{T}} = \overline{P}\,\overline{\overline{T}} = \overline{P}\,T$

(8) $P0 = 0$

(9) $P\overline{0} = \overline{0}$

(10) $(P\overline{S}) \odot T = \overline{S} \odot (PT)$

for P, Q in CS_r, for α,β in C, and for T,S in $U_2^{\otimes r}$.

Proof: Properties (1) to (9) are easily proved. Property (10) is proved as follows.

First, let $\bar{S} = \bar{u}_1\bar{u}_2\cdots\bar{u}_r$ and $T = v_1v_2\cdots v_r$, then

$$\bar{S} \odot (\hat{p}\ T)$$

$$= (\bar{u}_1\bar{u}_2\cdots\bar{u}_r) \odot (v_{p1}v_{p2}\cdots v_{pr})$$

$$= (\bar{u}_1 \circ v_{p1})(\bar{u}_2 \circ v_{p2})\cdots(\bar{u}_r \circ v_{pr})$$

Rearranging the factors and changing the index notation gives

$$\bar{S} \odot (\hat{p}\ T)$$

$$= (\bar{u}_{\hat{p}1} \circ v_1)(\bar{u}_{\hat{p}2} \circ v_2)\cdots(\bar{u}_{pr} \circ v_r)$$

$$= (\bar{u}_{\hat{p}1}\bar{u}_{\hat{p}2}\cdots\bar{u}_{\hat{p}r}) \odot (v_1v_2\cdots v_r)$$

$$= (p(\bar{u}_1\bar{u}_2\cdots\bar{u}_r)) \odot (v_1v_2\cdots v_r)$$

$$= (p\ \bar{S}) \odot T$$

This result is extended linearly to obtain

$$\bar{S} \odot (\hat{p}\ T) = (p\ \bar{S}) \odot T$$

for arbitrary $\bar{S}$ and T.

Hence

$$(P\bar{S}) \odot T$$

$$= (\sum \alpha_p (p\bar{S})) \odot T$$

$$= \sum \alpha_p ((p\bar{S}) \odot T)$$

$$= \sum \alpha_p (\bar{S} \odot (\hat{p}\ T))$$

$$= \bar{S} \odot (\hat{P}T)$$

Q.E.D.

Antisymmetrization and Symmetrization Operators in CS_r

The antisymmetrization operator $A(h,k)$ on h,k where $1 \leq h < k \leq r$, is defined as

$$A(h,k) = 1-(hk) .$$

The antisymmetrization operator $A(h,k,\ell)$ on h,k,ℓ where $1 \leq h < k < \ell \leq r$, is defined as

$$A(h,k,\ell) = \sum_{pi = i, \text{ if } i \neq h,k,\ell} \operatorname{sqn}(p)p = 1-(hk)-(h\ell)-(k\ell)+(hk\ell)+(h\ell k)$$

where $\operatorname{sqn}(p)=1$, if p is an even permutation, and -1, if p is an odd permutation.

The symmetrization operator $S(h_1,h_2,\cdots,h_s)$ on $h_1,h_2,\cdots,h_s$ where $1 \leq h_1 < h_2 < \cdot \cdot \cdot < h_s \leq r$, is defined as

$$S(h_1,h_2,\cdots,h_s) = \sum_{p\varepsilon H} p$$

where H is the subset of all elements in S_r that permute $h_1,h_2,\cdots,h_s$ only.

Theorem 1:

(1) $(hk)A(h,k) = A(h,k)(hk) = -A(h,k)$

(2) $A(h,k)A(h,k) = 2A(h,k)$

(3) $A(h,k,\ell)(hk) = A(h,k,\ell)(k\ell) = A(h,k,\ell)(h\ell) = -A(h,k,\ell)$

(4) $A(h,k,\ell)(hk\ell) = A(h,k,\ell)(k\ell h) = A(h,k,\ell)(\ell hk) = A(h,k,\ell)$

(5) $A(h,k,\ell)A(h,k) = A(h,k,\ell)A(k,\ell) = A(h,k,\ell)A(h,\ell) = 2A(h,k,\ell)$

(6) $A(h,k,\ell)A(h,k,\ell) = 6A(h,k,\ell)$

(7) $pS(h_1,h_2,\cdots,h_s) = S(h_1,h_2,\cdots,h_s)p = S(h_1,h_2,\cdots,h_s)$

for p in H.

$$(8)\ S(i_1,i_2,\cdots,i_k)S(h_1,h_2,\cdots,h_s) = S(h_1,h_2,\cdots,h_s)S(i_1,i_2,\cdots,i_k)$$
$$= k!S(h_1,h_2,\cdots,h_s)$$

if $\{i_1,i_2,\cdots,i_k\}\subset\{h_1,h_2,\cdots,h_s\}$.

$$(9)\ S(h_1,h_2,\cdots,h_s)S(h_1,h_2,\cdots,h_s) = s!S(h_1,h_2,\cdots,h_s)$$

$$(10)\ S(i,j)A(i,j) = A(i,j)S(i,j) = 0$$

$$(11)\ A(i,j)S(h_1,h_2,\cdots,h_s) = S(h_1,h_2,\cdots,h_s)A(i,j) = 0$$

if $\{i,j\}\subset\{h_1,h_2,\cdots,h_s\}$.

$$(12)\ A(h,k,\ell)S(k,\ell) = S(k,\ell)A(h,k,\ell) = 0$$

Theorem 2: The operator $A(h,k,\ell)$ acting on every T in $U_2^{\otimes r}$ always gives

$$A(h,k,\ell)T = 0$$

i.e.

$$A(h,k,\ell) = 0$$

as an operator equation.

Proof: First, we shall prove that if $S = A(1,2,3)(u_1u_2u_3\cdots u_r)$ for arbitrary $u_1,u_2,u_3,\cdots,u_r$ in U_2, then S=0. If $u_1=0$, the result is immediate. If $u_1 \neq 0$ and u_1,u_2 are dependent, then $u_2 = \alpha u_1$ for some α. Using the equation

$$\tfrac{1}{2}S(1,2)(u_1u_1u_3\cdots u_r) = u_1u_1u_3\cdots u_r$$

we obtain

$$S = \alpha A(1,2,3)(u_1u_1u_3\cdots u_r) = \tfrac{1}{2}\alpha A(1,2,3)S(1,2)(u_1u_1u_3\cdots u_r)$$
$$= 0$$

since $A(1,2,3)S(1,2) = 0$. If u_1,u_2 are independent, then $u_3 = \alpha u_1+\beta u_2$ for some α,β and

$$S = A(1,2,3)(\alpha u_1 u_2 u_1 \cdots u_r + \beta u_1 u_2 u_2 \cdots u_r)$$

$$= A(1,2,3)[\tfrac{1}{2}\alpha \mathcal{S}(1,3)(u_1 u_2 u_1 \cdots u_r) + \tfrac{1}{2}\beta \mathcal{S}(2,3)(u_1 u_2 u_2 \cdots u_r)]$$

$$= 0$$

since $A(1,2,3)\mathcal{S}(1,3) = 0$ and $A(1,2,3)\mathcal{S}(2,3) = 0$.

Now let p be any permutation in S_r such that $ph=1$, $pk=2$, $p\ell=3$. Then

$$A(h,k,\ell)(u_1 u_2 u_3 \cdots u_r) = \hat{p}pA(h,k,\ell)(u_1 u_2 u_3 \cdots u_r)$$

$$= \hat{p}A(1,2,3)p(u_1 u_2 u_3 \cdots u_r)$$

$$= \hat{p}A(1,2,3)u_{\hat{p}1} u_{\hat{p}2} u_{\hat{p}3} \cdots u_{\hat{p}r}$$

$$= \hat{p}0 = 0$$

The conjugate operation on this result also gives

$$A(h,k,\ell)(\bar{u}_1 \bar{u}_2 \bar{u}_3 \cdots \bar{u}_r) = 0$$

Finally we have

$$(A(h,k,\ell)T \mid \bar{u}_1, \bar{u}_2, \bar{u}_3, \cdots, \bar{u}_r)$$

$$= [A(h,k,\ell)T] \odot (\bar{u}_1 \bar{u}_2 \bar{u}_3 \cdots \bar{u}_r)$$

$$= T \odot [A(h,k,\ell)(\bar{u}_1 \bar{u}_2 \bar{u}_3 \cdots \bar{u}_r)]$$

$$= 0$$

Since $\bar{u}_1, \bar{u}_2, \bar{u}_3, \cdots, \bar{u}_r$ are arbitrary, then

$$A(h,k,\ell)T = 0$$ Q.E.D.

Corollary 1:

(1) $A(h,k)(hi)A(h,k) = A(h,k)$ if $i \neq h,k$

(2) $A(h,k)(jk)A(h,k) = A(h,k)$ if $j \neq h,k$

Corollary 2:

(1) $A(h,k)(hk\ell)A(h,k) = A(h,k)(k\ell h)A(h,k)$

$= A(h,k)(\ell hk)A(h,k) = -A(h,k)$

(2) $A(h,k)(\ell hk)A(h,k) = (m\cdots t)A(h,k)$

if $m,\cdots,t \neq h,k$.

(3) $A(h,k)(hm)(kn)A(h,k) = A(m,n)A(h,k)$

if $m \neq n$ and $m,n \neq h,k$.

In the useful special case of the symmetrization on h, h+1, h+2,$\cdots$, h+n=k, we shall abbreviate the notation to $\mathcal{S}(h\cdots k)$ in the following.

Subspaces of Antisymmetric and Symmetric Tensors

The space $(U_2^{\otimes 2})_A$ is a subspace of $U_2^{\otimes 2}$ consisting of antisymmetric tensors, i.e., all tensors in $U_2^{\otimes 2}$ which satisfy the property

$$(12)\ T = -T$$

The operator $\frac{1}{2}A(1,2)$ is a projection operator on $(U_2^{\otimes 2})_A$, thus

Theorem 1:

$$\frac{1}{2}A(1,2)T = T$$

iff T belongs to $(U_2^{\otimes 2})_A$.

Proof: If $T\varepsilon(U_2^{\otimes 2})_A$, then $(12)T = -T$, i.e., $-(12)T = T$. Adding T to both sides, we obtain $T-(12)T = [1-(12)]T = A(1,2)T = 2T$. Thus

$$\frac{1}{2}A(1,2)T = T$$

The converse is similar. Q.E.D.

Theorem 2: The subspace $(U_2^{\otimes 2})_A$ is one dimensional. We can choose the tensor

$$\Lambda = g_1g_2-g_2g_1$$

as a basis.

Proof: Let $T = \sum_{i,j=1}^{2} \alpha_{ij}g_ig_j$, with α_{ij} complex numbers. Then

$$-\sum_{i,j=1}^{2} \alpha_{ij}g_ig_j = -T = (12)T = (12)\sum_{i,j=1}^{2} \alpha_{ij}g_ig_j$$

$$= \sum_{i,j=1}^{2} \alpha_{ij}g_jg_i$$

imply

$$\alpha_{11} = \alpha_{22} = 0, \text{ and } \alpha_{21} = -\alpha_{12}$$

hence

$$T = \alpha_{12}(g_1g_2 - g_2g_1)$$

for all T in $(U_2^{\otimes 2})_A$. Thus, we can choose

$$\Lambda = g_1g_2-g_2g_1$$

as a basis, and $(U_2^{\otimes 2})_A$ is one dimensional. Q.E.D.

Similarly, the space $(\overline{U}_2^{\otimes 2})_A$ is a subspace of $\overline{U}_2^{\otimes 2}$, consisting of antisymmetric tensors. The operator $\frac{1}{2}A(1,2)$ is a projection operator on $(\overline{U}_2^{\otimes 2})_A$, the subspace $(\overline{U}_2^{\otimes 2})_A$ is one dimensional, and we can choose

$$\overline{\Lambda} = \overline{g}_1\overline{g}_2 - \overline{g}_2\overline{g}_1$$

as a basis.

The space $L_r \equiv (U_2^{\otimes r})_S$ is a subspace of $U_2^{\otimes r}$ consisting of symmetric tensors, i.e., all tensors in $U_2^{\otimes r}$ which have the property

$$pT = T$$

for every p in S_r. The projection operator on this subspace is

$\frac{1}{r!} S(1\cdots r)T = T$, thus

Theorem 3:

$$\frac{1}{r!} S(1\cdots r)T = T$$

iff $T \varepsilon L_r$.

Proof: If $T\varepsilon L_r$, then pT=T, for every p in S_r, hence

$$S(1\cdots r)T = \sum_{p\varepsilon S_r} pT = \sum_{p\varepsilon S_r} T = (r!)T$$

since the order of S_r is r!.

Conversely, if $\frac{1}{r!} S(1\cdots r)T=T$ for T in $U_2^{\otimes r}$, then

$$pT = p(\frac{1}{r!} S(1\cdots r)T) = \frac{1}{r!} pS(1\cdots r)T$$

$$= \frac{1}{r!} S(1\cdots r)T = T$$

since $pS(1\cdots r) = S(1\cdots r)$ for all p in S_r. Q.E.D.

Theorem 4: The subspace L_r has dimension r+1.

Proof: Each basis tensor $g_{a_1}g_{a_2}\cdots g_{a_r}$ of $U_2^{\otimes r}$ is projected by the operator $\frac{1}{r!} S(1\cdots r)$ onto one of the independent set of tensors $\frac{\alpha!(r-\alpha)!}{r!} S(1\cdots r)g_1^{\otimes\alpha} \times g_2^{\otimes(r-\alpha)}$ where $\alpha = 0,1,2,\cdots,r$. Q.E.D.

Similarly, the space $\bar{L}_r \equiv (\bar{U}_2^{\otimes r})_s$ is a subspace of $\bar{U}_2^{\otimes r}$ consisting of symmetric tensors. The operator $\frac{1}{r!} S(1\cdots r)$ is a projection operator on $\bar{L}_r$, $\frac{1}{r!} S(1\cdots r)\bar{T} = \bar{T}$ iff $\bar{T} \varepsilon \bar{L}_r$ and $\bar{L}_r$ has dimension r+1.

Linear Transformations on U_2 and $\overline{U}_2$

A linear transformation on U_2 is any map M which associates with each u in U_2 another element Mu in U_2 such that

$$M(\alpha u+\beta v) = \alpha(Mu) + \beta(Mv)$$

for each α,β in C and u,v in U_2. The unit transformation E is defined as

$$Eu = u$$

for every u in U_2. The sum M+N of two linear transformations M and N is another linear transformation defined as

$$(M+N)u = Mu + Nu$$

for every u in U_2, and the product αM of α in C with a linear transformation M is another linear transformation defined as

$$(\alpha M)u = \alpha(Mu)$$

for every u in U_2. The composition MN of two linear transformations M and N is a linear transformation defined as

$$(MN)u = M(Nu)$$

for every u in U_2. The inverse M^{-1}, if it exists, satisfies

$$M^{-1}M = MM^{-1} = E$$

The Hermitian adjoint M^+ of M is another linear transformation defined by the equation

$$\langle v|M^+u\rangle = \langle Mv|u\rangle$$

for all u,v in U_2. A transformation M is said to be unitary iff $M^+=M^{-1}$.

A transformation M is said to be Hermitian iff $M^+=M$, and anti-Hermitian iff $M^+=-M$.

A linear transformation L on $\bar{U}_2$ associates an element $L\bar{u}$ in $\bar{U}_2$ with each $\bar{u}$ in $\bar{U}_2$. Given a linear transformation M on U_2, its conjugate $\bar{M}$ and its transpose $\tilde{M} \equiv M^{\sim}$ are linear transformations on $\bar{U}_2$ defined by

$$\bar{M}\,\bar{u} = \overline{Mu}$$

$$(\tilde{M}\,\bar{v})\circ u = \bar{v} \circ (Mu)$$

respectively for all u and v in U_2. Also, $\bar{E}$ is the unit transformation on $\bar{U}_2$.

Theorem 1: $M^+ = \tilde{\bar{M}} = \overline{\tilde{M}}$ for any linear transformation M on U_2.

Proof: $(\tilde{\bar{M}}\,v)\circ\bar{u} = v \circ (\bar{M}\,\bar{u}) = v \circ (\overline{Mu})$

$$= \overline{\bar{v}\circ(Mu)} = \overline{(\tilde{M}\,\bar{v})\circ u} = (\overline{\tilde{M}}\,\bar{\bar{v}})\circ\bar{u} = (\overline{\tilde{M}}\,v)\circ\bar{u}$$

for all u,v in U_2, hence $\tilde{\bar{M}} = \overline{\tilde{M}}$.

$$\langle v|M^+u\rangle = \langle Mv|u\rangle = \overline{\langle u|Mv\rangle} = \overline{\bar{u}\circ(Mv)}$$

$$= \overline{(\tilde{M}\,\bar{u})\circ v} = (\overline{\tilde{M}}\,\bar{\bar{u}})\circ\bar{v} = (\overline{\tilde{M}}\,u)\circ\bar{v} = \bar{v}\circ(\overline{\tilde{M}}\,u)$$

$$= \langle v|\overline{\tilde{M}}\,u\rangle$$

for all u,v in U_2, hence $M^+ = \overline{\tilde{M}}$. Q.E.D.

The Linear Transformations $\Gamma(M)$ and $D(M)$ Acting on $U_2^{\otimes r}$ Corresponding to M

Given a linear transformation M on U_2, the linear transformations $\Gamma(M)$, $\Gamma_1(M)$, $\Gamma_2(M)$,$\cdots$,$\Gamma_r(M)$ on an arbitrary tensor T in $U_2^{\otimes r}$ for each r are defined as follows

$$(\Gamma(M)T \mid \bar{s}_1,\bar{s}_2,\cdots,\bar{s}_r) = (T \mid \widetilde{M}\bar{s}_1,\widetilde{M}\bar{s}_2,\cdots,\widetilde{M}\bar{s}_r)$$

$$(\Gamma_1(M)T \mid \bar{s}_1,\bar{s}_2,\cdots,\bar{s}_r) = (T \mid \widetilde{M}\bar{s}_1,\bar{s}_2,\cdots,\bar{s}_r)$$

$$(\Gamma_2(M)T \mid \bar{s}_1,\bar{s}_2,\cdots,\bar{s}_r) = (T \mid \bar{s}_1,\widetilde{M}\bar{s}_2,\bar{s}_3,\cdots,\bar{s}_r)$$

$$\cdots$$

$$(\Gamma_r(M)T \mid \bar{s}_1,\bar{s}_2,\cdots,\bar{s}_r) = (T \mid \bar{s}_1,\bar{s}_2,\cdots,\bar{s}_{r-1},\widetilde{M}\bar{s}_r)$$

for arbitrary $\bar{s}_1,\bar{s}_2,\cdots,\bar{s}_r$ in $\bar{U}_2$. It follows that in the special case in which T is a tensor product $u_1u_2\cdots u_r$, we have

$$\Gamma(M)(u_1u_2\cdots u_r) = (Mu_1)(Mu_2)\cdots(Mu_r)$$

$$\Gamma_1(M)(u_1u_2\cdots u_r) = (Mu_1)u_2\cdots u_r$$

$$\Gamma_2(M)(u_1u_2\cdots u_r) = u_1(Mu_2)u_3\cdots u_r$$

$$\cdots$$

$$\Gamma_r(M)(u_1u_2\cdots u_r) = u_1u_2\cdots u_{r-1}(Mu_r)$$

For $\lambda \in U_2^{\otimes 0} \equiv C$, we define

$$\Gamma(M)\lambda = \lambda$$

Theorem 1:

(1) $\Gamma(M) = \Gamma_1(M)\Gamma_2(M)\cdots\Gamma_r(M)$ when operating on $U_2^{\otimes r}$.

(2) $\Gamma(E) = \Gamma_1(E) = \Gamma_2(E) = \cdots = \Gamma_r(E)$

(3) $\Gamma(E)T = T$ for each T in $U_2^{\otimes r}$, i.e., $\Gamma(E)$ is the unit linear transformation on $U_2^{\otimes r}$ for each r.

(4) $\Gamma_k(\alpha M+\beta N) = \alpha\Gamma_k(M) + \beta\Gamma_k(N)$ for $k=1,2,\cdots,r$.

(5) $\Gamma_k(MN) = \Gamma_k(M)\Gamma_k(N)$ for $k=1,2,\cdots,r$.

(6) $\Gamma(MN) = \Gamma(M)\Gamma(N)$

for complex numbers α,β and linear transformation M and N on U_2.

Proof: (1) By definition of product of linear transformations $\Gamma_k(M)$ for $k=1,2,\cdots,r$; (2) and (3) are trivial; (4) is by definition of $\Gamma_k(M)$ and $\Gamma(M)$; for (5):

$$(\Gamma_k(MN)T \mid \bar{s}_1,\bar{s}_2,\cdots,\bar{s}_r) = (T \mid \bar{s}_1,\cdots,(\tilde{M}\tilde{N})\bar{s}_k,\cdots,\bar{s}_r)$$

$$= (T \mid \bar{s}_1,\cdots,\tilde{N}(\tilde{M}\bar{s}_k),\cdots,\bar{s}_r) = (\Gamma_k(N)T \mid \bar{s}_1,\cdots,\tilde{M}\bar{s}_k,\cdots,\bar{s}_r)$$

$$= (\Gamma_k(M)\Gamma_k(N)T \mid \bar{s}_1,\bar{s}_2,\cdots,\bar{s}_r)$$

for all $\bar{s}_1,\bar{s}_2,\cdots,\bar{s}_r$ in $\bar{U}_2$, hence

$$\Gamma_k(MN) = \Gamma_k(M)\Gamma_k(N)$$

The proof of (6) is similar. Q.E.D.

For each linear transformation M on U_2, define the linear transformation $D(M)$ on an arbitrary tensor T in $U_2^{\otimes r}$ for each r as

$$D(M)T = \sum_{k=1}^{r} \Gamma_k(M)T$$

i.e.

$$D(M) = \sum_{k=1}^{r} \Gamma_k(M)$$

when operating on $U_2^{\otimes r}$. It follows that

$$D(\alpha M+\beta N) = \alpha D(M) + \beta D(N)$$

for complex numbers α,β and linear transformations M and N on U_2. For $\lambda \in U_2^{\otimes 0} \equiv C$, we define

$$D(M)\lambda = 0$$

Some more useful properties of $\Gamma(M)$ and $D(M)$ are

Theorem 2:

$$(1)\ \Gamma(M)(T \otimes S) = [\Gamma(M)T] \otimes [\Gamma(M)S]$$

$$(2)\ D(M)(T \otimes S) = [D(M)T] \otimes S + T \otimes [D(M)S]$$

for T in $U_2^{\otimes r}$ and S in $U_2^{\otimes s}$ for each r and s.

Proof: (1) For all $\bar{s}_1, \cdots, \bar{s}_r, \bar{s}_{r+1}, \cdots, \bar{s}_{r+s}$ in $\bar{U}_2$

$$(T(M)(T \otimes S) \mid \bar{s}_1, \cdots, \bar{s}_r, \bar{s}_{r+1}, \cdots, \bar{s}_{r+s})$$

$$= (T \otimes S \mid \tilde{M}\bar{s}_1, \cdots, \tilde{M}\bar{s}_r, \tilde{M}\bar{s}_{r+1}, \cdots, \tilde{M}\bar{s}_{r+s})$$

$$= (T \mid \tilde{M}\bar{s}_1, \cdots, \tilde{M}\bar{s}_r)(S \mid \tilde{M}\bar{s}_{r+1}, \cdots, \tilde{M}\bar{s}_{r+s})$$

$$= (\Gamma(M)T \mid \bar{s}_1, \cdots, \bar{s}_r)(\Gamma(M)S \mid \bar{s}_{r+1}, \cdots, \bar{s}_{r+s})$$

$$= ([\Gamma(M)T] \otimes [\Gamma(M)S] \mid \bar{s}_1, \cdots, \bar{s}_r, \bar{s}_{r+1}, \cdots, \bar{s}_{r+s})$$

hence

$$\Gamma(M)(T \otimes S) = [\Gamma(M)T] \otimes [\Gamma(M)S]$$

For (2):

$$(D(M)(T \otimes S) \mid \bar{s}_1, \cdots, \bar{s}_{r+1}, \cdots, \bar{s}_{r+s})$$

$$= (\sum_{k=1}^{r+s} \Gamma_k(M)(T \otimes S) \mid \bar{s}_1, \cdots, \bar{s}_r, \bar{s}_{r+1}, \cdots, \bar{s}_{r+s})$$

$$= \sum_{k=1}^{r+s} (\Gamma_k(M)(T \otimes S) \mid \bar{s}_1, \cdots, \bar{s}_r, \bar{s}_{r+1}, \cdots, \bar{s}_{r+s})$$

$$= [(T \otimes S \mid \tilde{M}\bar{s}_1, \bar{s}_2, \cdots, \bar{s}_r, \bar{s}_{r+1}, \cdots, \bar{s}_{r+s})$$

$$+ (T \otimes S \mid \bar{s}_1, \tilde{M}\bar{s}_2, \cdots, \bar{s}_r, \bar{s}_{r+1}, \cdots, \bar{s}_{r+s}) + \cdots :$$

$$+ (T \otimes S \mid \bar{s}_1, \cdots, \bar{s}_{r-1}, \tilde{M}\bar{s}_r, \bar{s}_{r+1}, \cdots, \bar{s}_{r+s})]$$

$$+ [(T \otimes S \mid \bar{s}_1, \cdots, \bar{s}_r, \tilde{M}\bar{s}_{r+1}, \bar{s}_{r+2}, \cdots, \bar{s}_{r+s}) + \cdots$$

$$+ (T \otimes S \mid \bar{s}_1, \cdots, \bar{s}_r, \cdots, \bar{s}_{r+s-1}, \tilde{M}\bar{s}_{r+s})]$$

$$= [(T \mid \widetilde{Ms}_1, \bar{s}_2, \cdots, \bar{s}_r) + (T \mid \bar{s}_1, \widetilde{Ms}_2, \bar{s}_3, \cdots, \bar{s}_r) + \cdots$$

$$+ (T \mid \bar{s}_1, \cdots, \bar{s}_{r-1}, \widetilde{Ms}_r)]\ (S \mid \bar{s}_{r+1}, \cdots, \bar{s}_{r+s}) + (T \mid \bar{s}_1, \cdots, \bar{s}_r)$$

$$[(S \mid \widetilde{Ms}_{r+1}, \bar{s}_{r+2}, \cdots, \bar{s}_{r+s}) + \cdots + (S \mid \bar{s}_{r+1}, \cdots, \bar{s}_{r+s-1}, \widetilde{Ms}_{r+s})]$$

$$= [(\sum_{k=1}^{r} \Gamma_k(M)T \mid \bar{s}_1, \bar{s}_2, \cdots, \bar{s}_r)]\ (S \mid \bar{s}_{r+1}, \cdots, \bar{s}_{r+s})$$

$$+ (T \mid \bar{s}_1, \cdots, \bar{s}_r)\ [(\sum_{k=1}^{s} \Gamma_k(M)S \mid \bar{s}_{r+1}, \bar{s}_{r+2}, \cdots, \bar{s}_{r+s})]$$

$$= \{([\sum_{k=1}^{r} \Gamma_k(M)T] \otimes S \mid \bar{s}_1, \cdots, \bar{s}_{r+s}) \ + \ (T \otimes [\sum_{k=r+1}^{r+s} \Gamma_k(M)S] \mid \bar{s}_1, \cdots, \bar{s}_{r+s})\}$$

$$= ([D(M)T] \otimes S \mid \bar{s}_1, \cdots, \bar{s}_{r+s}) + (T \otimes [D(M)S] \mid \bar{s}_1, \cdots, \bar{s}_{r+s})$$

$$= ([D(M)T] \otimes S + T \otimes [D(M)S] \mid \bar{s}_1, \cdots, \bar{s}_{r+s})$$

hence

$$D(M)(T \otimes S) = [D(M)T] \otimes S + T \otimes [D(M)S] \qquad \text{Q.E.D.}$$

Theorem 3:

(1) $p[\Gamma_k(M)T] = \Gamma_k(M)(pT)$ for $k = 1,2,\cdots,r$.

(2) $p[\Gamma(M)T] = \Gamma(M)(pT)$

(3) $p[D(M)T] = D(M)(pT)$

(4) $\Gamma(M)\mathcal{S}(1\cdots r)T = \mathcal{S}(1\cdots r)\Gamma(M)T$

(5) $D(M)\mathcal{S}(1\cdots r)T = \mathcal{S}(1\cdots r)D(M)T$

for T in $U_2^{\otimes r}$ and p in S_r .

Proof: For $\bar{s}_1, \bar{s}_2, \cdots, \bar{s}_r$ in $\bar{U}_2$, we have

$$(p\Gamma_k(M)T \mid \bar{s}_1, \bar{s}_2, \cdots, \bar{s}_r)$$

$$= (\Gamma_k(M)T \mid \bar{s}_{p1}, \bar{s}_{p2}, \cdots, \bar{s}_{pr})$$

$$= (T \mid \bar{s}_{p1}, \bar{s}_{p2}, \cdots, \widetilde{Ms}_{pk}, \cdots, \bar{s}_{pr})$$

$$= (pT \mid \bar{s}_1, \bar{s}_2, \cdots, \widetilde{Ms}_k, \cdots, \bar{s}_r)$$

$$= (\Gamma_k(M)(pT) \mid \bar{s}_1, \bar{s}_2, \cdots, \bar{s}_r)$$

This proves (1). Similarly for (2). For (3), use the definition of $D(M)$ and (1). For (4) and (5), use (2) and (3) respectively. Q.E.D.

Determinants, Traces and the Commutator

Using an arbitrary tensor Ω in the one dimensional subspace $(U_2^{\otimes 2})_A$, we can define the determinant $\det(M)$ of M and the trace $\mathrm{tr}(M)$ of M for any linear transformation M on U_2. Since

Theorem 1: $\Gamma(M)\Omega$ is in $(U_2^{\otimes 2})_A$, if Ω is in $(U_2^{\otimes 2})_A$.

Proof: $\Omega \varepsilon (U_2^{\otimes 2})_A$ iff

$$\frac{1}{2} A(1,2)\Omega = \Omega$$

Now

$$\Gamma(M)\Omega = \Gamma(M)(\frac{1}{2} A(1,2)\Omega) = \frac{1}{2} A(1,2)(\Gamma(M)\Omega)$$

since

$$p\Gamma(M) = \Gamma(M)p,$$

thus, $\Gamma(M)\Omega$ is in $(U_2^{\otimes 2})_A$. Q.E.D.

Corollary 1: $\Gamma(M)\Omega$ is proportional to Ω, iff Ω is in $(U_2^{\otimes 2})_A$.

Proof: Since Ω is in $(U_2^{\otimes 2})_A$, the one dimensional subspace of $U_2^{\otimes 2}$, with

$$\Lambda = g_1g_2 - g_2g_1$$

as a basis. Thus by Theorem 1, we have $\Gamma(M)\Omega$ and Ω both proportional to Λ, hence $\Gamma(M)\Omega$ is proportional to Ω. This proves the corollary. Q.E.D.

From the above corollary, we call that proportional constant; the determinant, and define by

$$\Gamma(M)\Omega = \det(M)\Omega$$

Since $D(M)\Omega$ is also in $(U_2^{\otimes 2})_A$, it must be proportional to Ω. We define $\mathrm{tr}(M)$ by the equation

$$D(M)\Omega = \mathrm{tr}(M)\Omega$$

For two linear transformations M and N on U_2, their commutator $[M,N]$ is another linear transformation on U_2 defined as

$$[M,N] = MN-NM$$

Similarly, we can define the commutator of linear transformations on $\overline{U}_2$. Hence

Theorem 2:

(1) $D([M,N]) = [D(M),D(N)]$

(2) $[M,N]^+ = [N^+,M^+]$

(3) $\overline{[M,N]} = [\overline{M},\overline{N}]$

(4) $[M,N]^\sim = [\widetilde{N},\widetilde{M}]$

Proof: (2), (3) and (4) are trivial. To prove (1) we claim that

$$[\Gamma_i(A),\Gamma_j(B)] = \begin{cases} 0 & \text{if } i\neq j \\ \Gamma_i([A,B]) & \text{if } i=j \end{cases}$$

For $i < j$, then

$$\begin{aligned}
&([\Gamma_i(A),\Gamma_j(B)]T \mid \bar{s}_1,\bar{s}_2,\cdots,\bar{s}_r) \\
&= ((\Gamma_i(A)\Gamma_j(B)-\Gamma_j(B)\Gamma_i(A))T \mid \bar{s}_1,\bar{s}_2,\cdots,\bar{s}_r) \\
&= (\Gamma_i(A)\Gamma_j(B)T \mid \bar{s}_1,\bar{s}_2,\cdots,\bar{s}_r) - (\Gamma_j(B)\Gamma_i(A)T \mid \bar{s}_1,\bar{s}_2,\cdots,\bar{s}_r) \\
&= (\Gamma_j(B)T \mid \bar{s}_1,\cdots,\widetilde{A\bar{s}}_i,\cdots,\bar{s}_j,\cdots,\bar{s}_r)-(\Gamma_i(A)T \mid \bar{s}_1,\cdots,\bar{s}_i,\cdots,\widetilde{B\bar{s}}_j,\cdots,\bar{s}_r) \\
&= (T|\bar{s}_1,\cdots,\widetilde{A\bar{s}}_i,\cdots,\widetilde{B\bar{s}}_j,\cdots,\bar{s}_r) - (T|\bar{s}_1,\cdots,\widetilde{A\bar{s}}_i,\cdots,\widetilde{B\bar{s}}_j,\cdots,\bar{s}_r) \\
&= 0
\end{aligned}$$

Similarly for $i > j$. For $i = j$, we have

$$[\Gamma_i(A),\Gamma_i(B)] = \Gamma_i(A)\Gamma_i(B) - \Gamma_i(B)\Gamma_i(A)$$

$$= \Gamma_i(AB)-\Gamma_i(BA) = \Gamma_i(AB-BA) = \Gamma_i([A,B])$$

Here we used the property $\Gamma_i(MN) = \Gamma_i(M)\Gamma_i(N)$.

Now $[D(A),\ D(B)] = [\sum_{k=1}^{r}\Gamma_k(A),\ \sum_{\ell=1}^{r}\Gamma_\ell(B)]$

$$= \sum_{k=1}^{r}\ \sum_{\ell=1}^{r}[\Gamma_k(A),\Gamma_\ell(B)] = \sum_{k=1}^{r}[\Gamma_k(A),\Gamma_k(B)]$$

$$= \sum_{k=1}^{r}\Gamma_k([A,B]) = D([A,B])$$ Q.E.D.

Infinitesimal Transformations on U_2 and Infinitesimal Generators

Let $M(\varepsilon)$ be a one-parameter family of linear transformations on U_2 with $M(o) = E$. Then for infinitesimal values of ε, $M(\varepsilon)$ is an infinitesimal transformation. The transformation

$$A = \frac{d}{d\varepsilon}\ M(\varepsilon)\Big|_{\varepsilon\ =\ 0}$$

is called an infinitesimal generator of a transformation. The infinitesimal transformation $M(\varepsilon)$ can be expressed to first order in ε as

$$M(\varepsilon) \approx E + \varepsilon A$$

Representation of Infinitesimal Transformations and Infinitesimal Generators as Operators on $U_2^{\otimes r}$

The infinitesimal transformation operating on $U_2^{\otimes r}$ which corresponds to $M(\varepsilon)$ is $\Gamma(M(\varepsilon))$. The infinitesimal generator operating on $U_2^{\otimes r}$ which corresponds to A is $\frac{d}{d\varepsilon}\ \Gamma(M(\varepsilon))\big|_{\varepsilon\ =0}$. We have

$$[\frac{d}{d\varepsilon}\,\Gamma(M)]\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}\,[\Gamma_1(M)\Gamma_2(M)\cdots\Gamma_r(M)]\Big|_{\varepsilon=0}$$

$$= \{[\Gamma_1(\frac{dM}{d\varepsilon})\Gamma_2(M)\cdots\Gamma_r(M)] + [\Gamma_1(M)\Gamma_2(\frac{dM}{d\varepsilon})\cdots\Gamma_r(M)]$$

$$+ \cdots +[\Gamma_1(M)\Gamma_2(M)\cdots\Gamma_r(\frac{dM}{d\varepsilon})]\}\Big|_{\varepsilon=0}$$

$$= \{[\Gamma_1(A)\Gamma_2(E)\cdots\Gamma_r(E)] + [\Gamma_1(E)\Gamma_2(A)\cdots\Gamma_r(E)]$$

$$+ \cdots +[\Gamma_1(E)\Gamma_2(E)\cdots\Gamma_r(A)]\}$$

$$= \sum_{k=1}^{r}\Gamma_k(A) = D(A)$$

since $\Gamma_k(E) = \Gamma(E)$ is the unit transformation for $k=1,2,\cdots,r$.

This result can be derived with the help of infinitesimals as follows

$$\Gamma(M) = \Gamma_1(M)\Gamma_2(M)\cdots\Gamma_r(M)$$

$$\approx \Gamma_1(E+\varepsilon A)\Gamma_2(E+\varepsilon A)\cdots\Gamma_r(E+\varepsilon A)$$

$$= [\Gamma_1(E)+\varepsilon\Gamma_1(A)][\Gamma_2(E)+\varepsilon\Gamma_2(A)]\cdots[\Gamma_r(E)+\varepsilon\Gamma_r(A)]$$

$$= \Gamma(E) + \varepsilon\sum_{k=1}^{r}\Gamma_k(A)$$

$$= \Gamma(E) + \varepsilon D(A)$$

Then $[\frac{d}{d\varepsilon}\,\Gamma(M)]\Big|_{\varepsilon=0} = D(A)$

since higher order terms drop out for $\varepsilon=0$.

Theorem 1: There is a relation of trace and determinant under the operator $\frac{d}{d\varepsilon}$, i.e.

$$\mathrm{tr}(A) = \mathrm{tr}[\frac{d}{d\varepsilon}\,M(\varepsilon)\Big|_{\varepsilon=0}] = [\frac{d}{d\varepsilon}\,\det M(\varepsilon)]\Big|_{\varepsilon=0}$$

Proof: $[\frac{d}{d\varepsilon} \det M(\varepsilon)]|_{\varepsilon=0}\Omega = [\frac{d}{d\varepsilon} \det M(\varepsilon)\Omega]|_{\varepsilon=0}$

$= [\frac{d}{d\varepsilon} \Gamma(M)\Omega]|_{\varepsilon=0} = [\frac{d}{d\varepsilon} \Gamma(M)]|_{\varepsilon=0}\Omega$

$= D(A)\Omega = tr(A)\Omega$

Therefore

$$tr(A) = [\frac{d}{d\varepsilon} \det M(\varepsilon)]|_{\varepsilon=0} \qquad \text{Q.E.D.}$$

The Lie Group SU(2) and its Lie Algebra su(2)

Let SU(2) be the set of all linear transformations on U_2 with the properties

(1) $\det(M) = 1$ (unimodularity)

(2) $M^+ = M^{-1}$ (unitary)

Theorem 1: SU(2) is a Lie group.

Proof: By the definition of a Lie group G;

(1) G is a group.

(2) G is an analytic manifold.

(3) The mapping $(a,b) \rightarrow ab$ of $G \times G$ into G is an analytic mapping.

We know that SU(2) is a group with the operation the composite of two linear transformations, i.e., if $\det(M)=1$ and $\det(N)=1$, then $\det(MN) = \det(M)\det(N)=1$. Also, $(MN)^+ = N^+M^+ = N^{-1}M^{-1} = (MN)^{-1}$, since $M^+ = M^{-1}$ and $N^+ = N^{-1}$.

We omit the proof of (2) and (3). Q.E.D.

Let $[su(2)]_{aH}$ be the set of all linear transformations on U_2 with the properties

(1) $A^+ = -A$ (anti-Hermiticity)

(2) $tr(A) = 0$ (tracelessness)

Theorem 2: $[su(2)]_{aH}$ is a Lie algebra over R with the Lie bracket operation being $[A,B] = AB-BA$.

Proof: First, $[su(2)]_{aH}$ is a vector space over R, since

$$(\alpha A+\beta B)^{+} = \alpha A^{+}+\beta B^{+} = -\alpha A-\beta B = -(\alpha A+\beta B)$$

$$tr(\alpha A+\beta B) = \alpha tr(A)+\beta tr(B) = 0+0 = 0$$

Secondly, if A,B are in $[su(2)]_{aH}$, then $[A,B]$ is in $[su(2)]_{aH}$:

$$[A,B]^{+} = (AB-BA)^{+} = (AB)^{+}-(BA)^{+} = B^{+}A^{+}-A^{+}B^{+}$$

$$= (-B)(-A)-(-A)(-B) = BA-AB = -[A,B]$$

and

$$tr([A,B]) = tr(AB-BA) = tr(AB)-tr(BA)$$

$$= tr(AB)-tr(AB) = 0$$

Finally, we have

(1) $[A,B]$ is bilinear.

(2) $[A,B] + [B,A] = 0$

(3) $[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$

so that the Jacobi-identity is satisfied. Q.E.D.

Theorem 3: The set of all infinitesimal generators of SU(2) coincides with the set $[su(2)]_{aH}$.

Proof: Given an arbitrary infinitesimal generator A of SU(2). By theorem 1 of the previous section, we have

$$tr(A) = [\frac{d}{d\varepsilon} \det M(\varepsilon)]\Big|_{\varepsilon=0} = [\frac{d}{d\varepsilon} 1]\Big|_{\varepsilon=0}=0$$

since $\det M(\varepsilon)=1$ for all ε.

Next, differentiating $(M(\varepsilon))^{+}M(\varepsilon) = E$ with respect to ε gives

$$\frac{d}{d\varepsilon}(M(\varepsilon))^+(M(\varepsilon) + (M(\varepsilon))^+(\frac{d}{d\varepsilon} M(\varepsilon)) = 0$$

and setting $\varepsilon = 0$, yields

$$A^+E + E^+A = A^+ + A = 0$$

which is the anti-Hermiticity property.

Conversely, for $B \varepsilon [su(2)]_{aH}$, need to prove that $B = [\frac{d}{d\varepsilon} X(\varepsilon)]\Big|_{\varepsilon=0}$, for some $X(\varepsilon)$ in SU(2).

Let $X(\varepsilon) = e^{\varepsilon B}$, we have

$$[\frac{d}{d\varepsilon} X(\varepsilon)]\Big|_{\varepsilon=0} = [\frac{d}{d\varepsilon} e^{\varepsilon B}]\Big|_{\varepsilon=0} = e^{\varepsilon B}B\Big|_{\varepsilon=0} = B$$

Also

(1) $\det X(\varepsilon) = \det(e^{\varepsilon B}) = e^{tr(\varepsilon B)} = e^{\varepsilon tr(B)} = e^{\varepsilon 0} = 1$

since $tr(B) = 0$.

(2) $[X(\varepsilon)]^+ = (e^{\varepsilon B})^+ = e^{(\varepsilon B)^+} = e^{\varepsilon B^+} = e^{\varepsilon(-B)}$

$= e^{-\varepsilon B} = (e^{\varepsilon B})^{-1} = [X(\varepsilon)]^{-1}$

hence $X(\varepsilon) \varepsilon$ SU(2) . Q.E.D.

Now define $[su(2)]_H$ as the set of all traceless, Hermitian linear transformations on U_2. Thus each element H in $[su(2)]_H$ satisfies

$$tr(H) = 0$$
$$H^+ = H$$

It follows similarly as Theorem 2 above that $[su(2)]_H$ is a Lie algebra over R with its Lie bracket operation being $i[H,K]$. The one-to-one correspondence; H in $[su(2)]_H \leftrightarrow A = iH$ in $[su(2)]_{aH}$, is an isomorphism of the Lie algebras $[su(2)]_H$ and $[su(2)]_{aH}$. From now on we shall need only the space $[su(2)]_H$, and will drop the subscript H, thus

$su(2) \equiv [su(2)]_H$. We can also define C[su(2)] as the set of all complex linear combinations of the form A+iB with A and B in su(2). This set C[su(2)] with the bracket operation i[H,K] for H,K in C[su(2)] , is also a Lie algebra.

The Space $[su(2)]^2$. The Γ^r, D^r, Δ^r Representations of SU(2), su(2), $[su(2)]^2$ Respectively

The Γ^r representation of SU(2) with the representation space $L_r = (U_2^{\otimes r})_s$ is the homomorphism ;$M \to \Gamma(M)$, for all M in SU(2) with the previously defined operator $\Gamma(M)$ restricted to the domain L_r. The D^r representation of su(2) with the representation space L_r is the homomorphism ;$A \to D(A)$, for all A in su(2) with the previously defined operator D(A) restricted to the domain L_r. The D^r representation of C[su(2)] with the representation space L_r is the homomorphism ;$H \to D(H)$, for all H in C[su(2)] with D(H) restricted to operate on the domain L_r.

The representation space L_r is a unitary space, therefore $\Gamma(M)^+$ and $D(A)^+$ can be defined for M in SU(2) and A in su(2). It can be shown that $\Gamma(M)^+ = \Gamma(M)^{-1}$ and $D(A)^+ = D(A)$ for all M in SU(2) and all A in su(2), thus Γ^r is called a unitary representation of SU(2) and D^r is called a Hermitian representation of su(2). A subspace W of L_r is called an invariant subspace if $T \varepsilon W$ implies $\Gamma(M) T \varepsilon W$ for all M in SU(2). Similarly for all M in su(2) or C[su(2)]. The representation is said to be irreducible if there are no invariant subspaces except L_r itself and the subspace {0} consisting of the zero element of L_r only. The representation Γ^r of SU(2) and D^r of su(2) and C[su(2)] are irreducible.

The set $[su(2)]^2$ of all operators expressible in the form $\sum_{k=1}^{\ell} A_k B_k$ with A_k, B_k in su(2) contains elements not in su(2). The Δ^r representation of $[su(2)]^2$ is defined by the map $\sum_{k=1}^{\ell} A_k B_k \to \Delta(\sum_{k=1}^{\ell} A_k B_k)$ where

$$\Delta(\sum_{k=1}^{\ell} A_k B_k) = \sum_{k=1}^{\ell} D(A_k)D(B_k)$$

and $\Delta(\sum_{k=1}^{\ell} A_k B_k)$ is restricted to operate on the domain L_r. Note that

Theorem 1: $\Delta([A,B]) = [D(A),D(B)] = D([A,B])$

Proof: $\Delta([A,B]) = \Delta(AB-BA)$

$$= D(A)D(B) - D(B)D(A) = [D(A),D(B)] \qquad \text{Q.E.D.}$$

Therefore elements in $[su(2)]^2$ which also belong to su(2) are represented in the Δ^r representation by the same operator which represents it in the D^r representation.

The Basis J_x, J_y, J_z of su(2) and the Operator $J^2 = J_x^2 + J_y^2 + J_z^2$ in $[su(2)]^2$

The operators J_x, J_y, J_z on U_2 are uniquely defined by the requirement that

$$J_x g_1 = \frac{1}{2} g_2 \;, \qquad J_x g_2 = \frac{1}{2} g_1$$

$$J_y g_1 = \frac{1}{2} i g_2 \;, \qquad J_y g_2 = -\frac{1}{2} i g_1$$

$$J_z g_1 = \frac{1}{2} g_1 \;, \qquad J_z g_2 = -\frac{1}{2} g_2$$

where g_1, g_2 is the orthonormal basis of U_2 previously given.

Theorem 1: J_x, J_y, J_z is a basis for su(2).

Theorem 2:

(1) $[J_x, J_y] = iJ_z$

(2) $[J_y, J_z] = iJ_x$

(3) $[J_z, J_x] = iJ_y$

Proof: To prove $[J_x,J_y] = iJ_z$, we need only to prove that they give the same result when acting on g_1 and g_2 respectively. Now

$$\begin{aligned}
[J_x,J_y]g_1 &= (J_xJ_y-J_yJ_x)g_1 \\
&= J_x(J_yg_1) - J_y(J_xg_1) = J_x(\tfrac{1}{2}\, ig_2)-J_y(\tfrac{1}{2}\, g_2) \\
&= \tfrac{1}{2}\, i(J_xg_2) - \tfrac{1}{2}\,(J_yg_2) = \tfrac{1}{2}\, i(\tfrac{1}{2}\, g_1) - \tfrac{1}{2}(-\tfrac{1}{2}\, ig_1) \\
&= \tfrac{1}{2}\, ig_1 = i(J_zg_1)
\end{aligned}$$

and

$$\begin{aligned}
[J_x,J_y]g_2 &= (J_xJ_y-J_yJ_x)g_2 \\
&= J_x(J_yg_2) - J_y(J_xg_2) = J_x(-\tfrac{1}{2}\, ig_1) - J_y(\tfrac{1}{2}\, g_1) \\
&= -\tfrac{1}{2}\, i(J_xg_1) - \tfrac{1}{2}(J_yg_1) = -\tfrac{1}{2}\, i(\tfrac{1}{2}\, g_2) - \tfrac{1}{2}(\tfrac{1}{2}\, ig_2) \\
&= -\tfrac{1}{2}\, ig_2 = i(J_zg_2)
\end{aligned}$$

Similarly for (2) and (3). Q.E.D.

Corollary 1:

(1) $[J_x,J_x] = [J_y,J_y] = [J_z,J_z] = 0$

(2) $[[J_x,J_y],J_z] = [[J_y,J_z],J_x] = [[J_z,J_x],J_y] = 0$

Thus the Jacobi-identity is satisfied.

Also define

$$J+ = J_x + iJ_y$$

$$J- = J_x - iJ_y$$

which are in C[su(2)]. The operator J^2 is defined as

$$J^2 = J_xJ_x + J_yJ_y + J_zJ_z$$

and is in $[su(2)]^2$.

The following relations are easily verified

(1) $J_+g_2 = g_1$, and $J_+g_1 = 0$

(2) $J_-g_1 = g_2$, and $J_-g_2 = 0$

(3) $[J_z,J_+] = J_+$, and $[J_z,J_-] = -J_-$

(4) $J^2 = \frac{1}{2}(J_+J_- + J_-J_+) + J_zJ_z$

(5) $[J_x,J^2] = [J_y,J^2] = [J_z,J^2] = 0$

(6) $J^2g_1 = \frac{3}{4}g_1$, and $J^2g_2 = \frac{3}{4}g_2$

The Orthogonal Basis $G(\alpha,r-\alpha)$ of L_r for $\alpha=0,1,2,\cdots,r$

In order to construct simultaneous eigenstates of the operators $\Delta(J^2)$ and $D(J_z)$, we need the quantities $G(\alpha,\beta)$, where $\alpha,\beta = 0,1,2,\cdots$, which are defined as

$$G(\alpha,\beta) = (\alpha!\beta!)^{-1}S(1\cdots\alpha+\beta)(g_1^{\otimes\alpha}\, g_2^{\otimes\beta})$$

$$= (\alpha!\beta!)^{-1}\sum_p p(g_1^{\otimes\alpha}g_2^{\otimes\beta})$$

where p goes over all permutations in $S_{\alpha+\beta}$. Note that there are $\alpha!\beta!$ permutations in $S_{\alpha+\beta}$ that leave $g_1^{\otimes\alpha}g_2^{\otimes\beta}$ invariant. Also, for each distinct monomial term, there are $\alpha!\beta!$ permutations that transform $g_1^{\otimes\alpha}g_2^{\otimes\beta}$ into this monomial term. Thus when terms are collected and divided by $\alpha!\beta!$, the resulting quantity $G(\alpha,\beta)$ contains each distinct monomial term with a coefficient of one.

Some special cases are

$$G(\alpha,0) = g_1^{\otimes\alpha}$$

$$G(0,\beta) = g_2^{\otimes\beta}$$

$$G(0,0) = 1$$

for $\alpha=1,2,\cdots$ and $\beta=1,2,\cdots$.

Theorem 1: For each r, the tensors $G(\alpha,r-\alpha)$ for $\alpha=0,1,2,\cdots,r$ form a basis for L_r .

Proof: Suppose $\sum_{\alpha=0}^{r} b_\alpha G(\alpha,r-\alpha)=0$, then for $\alpha'=0,1,\cdots,r$ we have

$$0 = (\bar{g}_1^{\otimes\alpha'} \otimes \bar{g}_2^{\otimes(r-\alpha')}) \odot [\sum_{\alpha=0}^{r} b_\alpha G(\alpha,r-\alpha)]$$

$$= (\alpha'!(r-\alpha')!)^{-1} b_{\alpha'}=0$$

Therefore $G(\alpha,r-\alpha)$ for $\alpha=0,1,\cdots,r$, are linearly independent.

Consider any T in L_r. Thus

$$T = \sum_{a_1,a_2,\cdots,a_r} T_{a_1a_2\cdots a_r} g_{a_1} g_{a_2} \cdots g_{a_r}$$

we have

$$T = \frac{1}{r!} S(1\cdots r)T = \frac{1}{r!} \sum_{a_1,a_2,\cdots,a_r} T_{a_1a_2\cdots a_r} S(1\cdots r) g_{a_1} g_{a_2} \cdots g_{a_r}$$

$$= \frac{1}{r!} \sum_{a_1,a_2,\cdots,a_r} T_{a_1a_2\cdots a_r} S(1\cdots r)\ g_1^{\otimes\alpha} g_2^{\otimes\beta}$$

with $\alpha= \sum_i \delta_{1a_i}$ = number of times g_1 appears, and $\beta = \sum_i \delta_{2a_i}$ = number of times g_2 appears.

Therefore the tensors $S(1\cdots r)\ g_1^{\otimes\alpha} g_2^{\otimes\beta}$ span L_r. Hence $G(\alpha,r-\alpha)$ for $\alpha=0,1,\cdots,r$ form a basis for L_r. Q.E.D.

Computation of $D(J_z)\ G(\alpha,\beta)$

First, for $\alpha=0,1,2,\cdots$, we have

$$D(J_z)g_1^{\otimes\alpha} = (J_zg_1)g_1^{\otimes(\alpha-1)} + g_1(J_zg_1)g_1^{\otimes(\alpha-2)} + g_1^{\otimes 2}(J_zg_1)g_1^{\otimes(\alpha-3)}$$

$$+ \cdots + g_1^{\otimes(\alpha-1)}\ (J_zg_1) = \frac{1}{2}\ \alpha g_1^{\otimes\alpha}$$

Similarly, for $\beta = 0,1,2,\cdots$.

$$D(J_z)\ g_2^{\otimes\beta} = -\ \frac{1}{2}\ \beta g_2^{\otimes\beta}$$

Then

$$D(J_z)(g_1^{\otimes\alpha}\ g_2^{\otimes\beta}) = [D(J_z)g_1^{\otimes\alpha}]g_2^{\otimes\beta} + g_1^{\otimes\alpha}[D(J_z)g_2^{\otimes\beta}]$$

$$= \frac{1}{2}\ (\alpha-\beta)\ g_1^{\otimes\alpha}\ g_2^{\otimes\beta}$$

and finally

$$D(J_z)G(\alpha,\beta) = (\alpha!\beta!)^{-1}S(1\cdots\alpha+\beta)D(J_z)(g_1^{\otimes\alpha}g_2^{\otimes\beta})$$

$$= \frac{1}{2}(\alpha-\beta)(\alpha!\beta!)^{-1}S(1\cdots\alpha+\beta)(g_1^{\otimes\alpha}g_2^{\otimes\beta})$$

$$= \frac{1}{2}(\alpha-\beta)G(\alpha,\beta)$$

Computation of $\Delta(J^2)G(\alpha,\beta)$

First, for $\beta = 1,2,\cdots$, we have

$$D(J_+)g_2^{\otimes\beta} = (J_+g_2)g_2^{\otimes(\beta-1)} + g_2(J_+g_2)g_2^{\otimes(\beta-2)} + \cdots +$$

$$g_2^{\otimes(\beta-1)}(J_+g_2) = g_1g_2^{\otimes(\beta-1)} + g_2g_1g_2^{\otimes(\beta-2)} + \cdots + g_2^{\otimes(\beta-1)}g_1$$

$$= [1+(12)+(13) + \cdots + (1\alpha)]g_1g_2^{\otimes(\beta-1)}$$

For $\beta = 0$, the result is zero. Similarly, for all α

$$D(J_+)g_1^{\otimes\alpha} = 0$$

Then we have

$$D(J_+)(g_1^{\otimes\alpha} g_2^{\otimes\beta}) = g_1^{\otimes\alpha}[1+(12)+(13) + \cdots + (1\beta)]g_1 g_2^{\otimes(\beta-1)}$$

$$= [1+(\alpha+1,\alpha+2)+(\alpha+1,\alpha+3) + \cdots + (\alpha+1,\alpha+\beta)]\ g_1^{\otimes(\alpha+1)} g_2^{\otimes(\beta-1)}$$

for $\beta \neq 0$, and the result is zero for $\beta=0$. Finally

$$D(J_+)G(\alpha,\beta) = (\alpha!\beta!)^{-1} S(1\cdots\alpha+\beta)D(J_+)(g_1^{\otimes\alpha} g_2^{\otimes\beta})$$

$$= (\alpha!\beta!)^{-1} S(1\cdots\alpha+\beta)[1+(\alpha+1,\alpha+2)+(\alpha+1,\alpha+3)+\cdots+(\alpha+1,\alpha+\beta)]g_1^{\otimes(\alpha+1)} g_2^{\otimes(\beta-1)}$$

$$= (\alpha!\beta!)^{-1}\beta S(1\cdots\alpha+\beta)\ g_1^{\otimes(\alpha+1)}\ g_2^{\otimes(\beta-1)}$$

$$= (\alpha+1)G(\alpha+1,\beta-1)$$

for $\beta \neq 0$, and is equal to zero when $\beta=0$.

Similarly

$$D(J_-)G(\alpha,\beta) = (\beta+1)G(\alpha-1,\beta+1)$$

for $\alpha \neq 0$, and is equal to zero when $\alpha=0$.

Using the equation

$$\Delta(J^2) = \frac{1}{2} D(J_+)D(J_-) + \frac{1}{2} D(J_-)D(J_+) + D(J_z)D(J_z)$$

it follows that

$$\Delta(J^2)G(\alpha,\beta)$$

$$= [\ \frac{1}{2}\alpha(\beta+1) + \frac{1}{2}\beta(\alpha+1) + \frac{1}{4}(\alpha-\beta)^2]G(\alpha,\beta)$$

$$= \frac{1}{2}(\alpha+\beta)[\ \frac{1}{2}(\alpha+\beta)+1]\ G(\alpha,\beta)$$

The tensor $G(\alpha,\beta)$ is in $L_{\alpha+\beta}$ and has $(\alpha+\beta)!\ (\alpha!\beta!)^{-1}$ orthonormal terms. Consequently

$$\langle G(\alpha,\beta)\,|\,G(\alpha,\beta)\rangle = (\alpha+\beta)!\ (\alpha!\beta!)^{-1}$$

The Orthonormal Basis of L_{2j} Consisting of Simultaneous Eigentensors $\Phi(j,m)$

Replacing α,β by j,m by

$$j = \frac{1}{2}(\alpha+\beta) , \quad m = \frac{1}{2}(\alpha-\beta)$$

and define

$$\Phi(j,m) = [j+m)!(j-m)!]^{\frac{1}{2}} [(2j)!]^{-\frac{1}{2}} G(j+m,j-m)$$

where the allowed values of j and m are

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$$

$$m = -j, -j+1, \cdots, 0, 1, \cdots, j-1, j$$

It follows that

(1) $<\Phi(j,m_1) \mid \Phi(j,m_2)> = \delta_{m_1m_2}$

(2) $\Delta(J^2)\Phi(j,m) = j(j+1)\Phi(j,m)$

(3) $D(J_z)\Phi(j,m) = m\Phi(j,m)$

Thus for each given allowed value of j, the tensors $\Phi(j,m)$ for all allowed values of m form an orthonormal basis of L_{2j}, are eigentensors of the operator $\Delta(J^2)$ with eigenvalue j(j+1), and are eigentensors of $D(J_z)$ with eigenvalue m.

The Kronecker Product Representations $\Gamma^r x\Gamma^s$, $D^r xD^s$ and $\Delta^r x\Delta^s$

Given two representations Γ^r and Γ^s of SU(2), D^r and D^s of su(2) (or C[su(2)]), and Δ^r and Δ^s of $[su(2)]^2$ with the representation spaces L_r and L_s respectively, the corresponding Kronecker product representations $\Gamma^r x\Gamma^s$, $D^r xD^s$ and $\Delta^r x\Delta^s$ all have the representation space $L_r \otimes L_s$, and are given by the homomorphisms ;$M \to \Gamma(M)$ for all M in $SU(2)$; $A \to D(A)$ for all A in $su(2)$, and $F \to \Delta(F)$ for all F in $[su(2)]^2$ respectively where the operators $\Gamma(M)$, $D(A)$ and $\Delta(F)$ are restricted to operate on the domain $L_r \otimes L_s$. The Kronecker product representation is reducible unless r or s equals zero since otherwise, L_r and L_s are the invariant subspaces. Also, the dimension of $L_r \otimes L_s$ is $(r+1)(s+1)$.

The projection operator onto the subspace $L_r \otimes L_s$ of $U_2^{\otimes(r+s)}$ is

$$(r!s!)^{-1} S(1\cdots r) S(r+1\cdots r+s)$$

Using $r=2j_1$, $s=2j_2$, we note that the tensors

$$\Phi(j_1,m_1;\ j_2,m_2) \equiv \Phi(j_1,m_1) \otimes \Phi(j_2,m_2)$$

for given values of j_1 and j_2 and all allowed values of m_1 and m_2, are an orthonormal basis of $L_{2j_1} \otimes L_{2j_2}$. It is proved as follows

$$\begin{aligned}
&\langle\Phi(j_1,m_1;j_2,m_2)|\ \Phi(j_1,m_3;j_2,m_4)\rangle \\
&= [\overline{\Phi}(j_1,m_1) \otimes \overline{\Phi}(j_2,m_2)] \odot [\Phi(j_1,m_3) \otimes \Phi(j_2,m_4)] \\
&= [\overline{\Phi}(j_1,m_1) \odot \Phi(j_1,m_3)]\ [\overline{\Phi}(j_2,m_2) \odot \Phi(j_2,m_4)] \\
&= \langle\Phi(j_1,m_1)|\ \Phi(j_1,m_3)\rangle\ \langle\Phi(j_2,m_2)|\ \Phi(j_2,m_4)\rangle \\
&= \delta_{m_1 m_3}\ \delta_{m_2 m_4}
\end{aligned}$$

This basis is a set of simultaneous eigentensors of operators which we now define:

For each A in $su(2)$, define

$$D_1(A) = \sum_{k=1}^{r} \Gamma_k(A)$$

$$D_2(A) = \sum_{k=r+1}^{r+s} \Gamma_k(A)$$

as operators on $U_2^{\otimes(r+s)}$. Hence

Theorem 1:

(1) $D(A) = D_1(A) + D_2(A)$

(2) $D_1(\alpha A+\beta B) = \alpha D_1(A) + \beta D_1(B)$

(3) $D_2(\alpha A+\beta B) = \alpha D_2(A) + \beta D_2(B)$

Theorem 2:

(1) $D_i(A)\mathcal{S}(1\cdots r) = \mathcal{S}(1\cdots r)\ D_i(A)$

(2) $D_i(A)\mathcal{S}(r+1\cdots r+s) = \mathcal{S}(r+1\cdots r+s)\ D_i(A)$

for $i = 1,2$.

Theorem 3:

(1) $D_1(A)\ (T \otimes S) = [D(A)T] \otimes S$

(2) $D_2(A)\ (T \otimes S) = T \otimes [D(A)S]$

for T in $U_2^{\otimes r}$ and S in $U_2^{\otimes s}$.

Proof: For $\bar{w}_1, \bar{w}_2, \cdots, \bar{w}_r, \bar{w}_{r+1}, \cdots, \bar{w}_{r+s}$ in $\bar{U}_2$

$$(D_1(A)(T \otimes S) \mid \bar{w}_1, \bar{w}_2, \cdots, \bar{w}_r, \bar{w}_{r+1}, \cdots, \bar{w}_{r+s})$$

$$= (\sum_{k=1}^{r} \Gamma_k(A)(T \otimes S) \mid \bar{w}_1, \bar{w}_2, \cdots, \bar{w}_r, \bar{w}_{r+1}, \cdots, \bar{w}_{r+s})$$

$$= \sum_{k=1}^{r} (\Gamma_k(A)(T \otimes S) \mid \bar{w}_1, \bar{w}_2, \cdots, \bar{w}_r, \bar{w}_{r+1}, \cdots, \bar{w}_{r+s})$$

$$= \sum_{k=1}^{r} (T \otimes S \mid \bar{w}_1, \cdots, \tilde{A}\bar{w}_k, \cdots, \bar{w}_r, \cdots, \bar{w}_{r+s})$$

$$= [\sum_{k=1}^{r} (T \mid \bar{w}_1, \cdots, \tilde{A}\bar{w}_k, \cdots, \bar{w}_r)]\ (S \mid \bar{w}_{r+1}, \cdots, \bar{w}_{r+s})$$

$$= (\sum_{k=1}^{r} \Gamma_k(A)T \mid \bar{w}_1, \cdots, \bar{w}_r)(S \mid \bar{w}_{r+1}, \cdots, \bar{w}_{r+s})$$

$$= (D(A)T \mid \bar{w}_1,\cdots,\bar{w}_r)\ (S \mid \bar{w}_{r+1},\cdots,\bar{w}_{r+s})$$

$$= ([D(A)T] \otimes S \mid \bar{w}_1,\cdots,\bar{w}_r,\cdots,\bar{w}_{r+s})$$

Hence $D_1(A)\ (T \otimes S) = [D(A)T] \otimes S$. Note that the D operator of this theorem is the D^r representation defined on L_r and is different from the D operator which is the $D^r x D^s$ representation defined on $L_r \otimes L_s$. Similarly for (2). Q.E.D.

For each $F = \sum_{k=1}^{\ell} A_k B_k$ in $[su(2)]^2$, define

$$\Delta_1\left(\sum_{k=1}^{\ell} A_k B_k\right) = \sum_{k=1}^{\ell} D_1(A_k) D_1(B_k)$$

$$\Delta_2\left(\sum_{k=1}^{\ell} A_k B_k\right) = \sum_{k=1}^{\ell} D_2(A_k) D_2(B_k)$$

as operators on $U_2^{\otimes(r+s)}$. Hence

Theorem 4:

(1) $\Delta_i(F)\mathcal{S}(1\cdots r) = \mathcal{S}(1\cdots r)\Delta_i(F)$

(2) $\Delta_i(F)\mathcal{S}(r+1\cdots r+s) = \mathcal{S}(r+1\cdots r+s)\Delta_i(F)$

for i=1,2.

(3) $\Delta_1(F)\ (T \otimes S) = [\Delta(F)T] \otimes S$

(4) $\Delta_2(F)\ (T \otimes S) = T \otimes [\Delta(F)S]$

for T in $U_2^{\otimes r}$ and S in $U_2^{\otimes s}$.

Note that

$$\Delta(F) \neq \Delta_1(F) + \Delta_2(F)$$

Now we have

$$\Delta_1(J^2)\Phi(j_1,m_1;j_2,m_2) = [\Delta(J^2)\Phi(j_1,m_1)] \otimes \Phi(j_2,m_2)$$

$$= j_1(j_1+1)\Phi(j_1,m_1;j_2,m_2)$$

$$\Delta_2(J^2)\Phi(j_1,m_1;j_2,m_2)=\Phi(j_1,m_1) \otimes [\Delta(J^2)\Phi(j_2,m_2)]$$

$$= j_2(j_2+1)\Phi(j_1,m_1;j_2,m_2)$$

$$D_1(J_z)\Phi(j_1,m_1;j_2,m_2) = [D(J_z)\Phi(j_1,m_1)] \otimes \Phi(j_2,m_2)$$

$$= m_1\Phi(j_1,m_1;j_2,m_2)$$

$$D_2(J_z)\Phi(j_1,m_1;j_2,m_1) = \Phi(j_1,m_1) \otimes [D(J_z)\Phi(j_2,m_2)]$$

$$= m_2\Phi(j_1,m_1;j_2,m_2)$$

Thus $\Phi(j_1,m_1;j_2,m_2)$ is a simultaneous eigentensor of $\Delta_1(J^2)$ with eigenvalue $j_1(j_1+1)$, of $\Delta_2(J^2)$ with eigenvalue $j_2(j_2+1)$, of $D_1(J_z)$ with eigenvalue m_1, and of $D_2(J_z)$ with eigenvalue m_2.

The Subspaces $L^t_{r,s}$ and the Direct Sum $L_r \otimes L_s = \bigoplus_{\substack{t=|r-s| \\ (\Delta t=2)}}^{r+s} L^t_{r,s}$

Now we construct the subspaces $L^t_{r,s}$ of $L_r \otimes L_s$, where $t = |r-s|,|r-s|+2,|r-s|+4,\cdots,r+s$. The subspace $L^t_{r,s}$ is defined as the set of all tensors in $L_r \otimes L_s$ which can be expressed in the form

$$S(1\cdots r)S(r+1\cdots r+s)\ p_{(q,t)}\Lambda^{\otimes q} \otimes T$$

with T in L_t, where $q = \frac{1}{2}(r+s-t)$, and where $p_{(q,t)}$ is a permutation in S_{r+s} defined as

$$p_{(q,t)}(2k-1) = k \qquad \text{for } k = 1,2,\cdots,q$$

$$p_{(q,t)}(2k) = q+t+k \qquad \text{for } k = 1,2,\cdots,q$$

$$p_{(q,t)}(2q+k) = q+k \qquad \text{for } k = 1,2,\cdots,t$$

For fixed r and s, the possible value of q are $q=0,1,2,\cdots,\min(r,s)$. For q values $\min(r,s) < q \leq \frac{1}{2}(r+s)$, the zero tensor is produced. The range of values for q and the relation $t=r+s-2q$ leads to the values of t, which are $t=|r-s|,|r-s|+2,|r-s|+4,\cdots,r+s$. The permutation $p_{(q,t)}$ on a tensor product acts as follows:

$$p_{(q,t)}\, u_1 u_2 \cdots u_{2q} u_{2q+1} u_{2q+2} \cdots u_{2q+t}$$

$$= u_1 u_3 \cdots u_{2q-1} u_{2q+1} u_{2q+2} \cdots u_{2q+t} u_2 u_4 \cdots u_{2q}$$

thus $p_{(q,t)}$ takes the vectors in the first q even numbered positions and moves them to the right end, keeping them in the same order with respect to each other.

The subspace $L^t_{r,s}$ has the same dimension as L_t, which is t+1. This will be shown later by explicit construction of a basis for $L^t_{r,s}$. Summing the dimensions gives

$$\sum_{\substack{t=|r-s| \\ (\Delta t=2)}}^{r+s} (t+1) = (r+1)(s+1)$$

which agrees with the dimension of $L_r \otimes L_s$. It will be shown later that the subspaces $L_{r,s}^{t}$ are orthogonal to one another, thus $L_r \otimes L_s$ is the direct sum of these subspaces, i.e.

$$L_r \otimes L_s = \bigoplus_{\substack{t=|r-s| \\ (\Delta t=2)}}^{r+s} L_{r,s}^{t}$$

Action of $\Gamma(M)$, $D(A)$, and $\Delta(F)$ on the Subspaces $L_{r,s}^{t}$

For M in SU(2), since det(M) = 1, we have

$$\Gamma(M)\Lambda = \Lambda$$

Then

$$\Gamma(M)\Lambda^{\otimes q} = [\Gamma(M)\Lambda]^{\otimes q} = \Lambda^{\otimes q}$$

and

$$\Gamma(M)(\Lambda^{\otimes q} \otimes T) = [\Gamma(M)\Lambda^{\otimes q}] \otimes [\Gamma(M)T]$$

$$= \Lambda^{\otimes q} \otimes [\Gamma(M)T]$$

for T in L_t. Since

$$\Gamma(M)P = P\Gamma(M)$$

for any P in CS_{r+s}, it follows that

$$\Gamma(M)[S(1\cdots r)S(r+1\cdots r+s)p_{(q,t)}\Lambda^{\otimes q} \otimes T]$$

$$= S(1\cdots r)S(r+1\cdots r+s)p_{(q,t)}\Lambda^{\otimes q} \otimes [\Gamma(M)T]$$

for T in L_t.

For A in su(2), since tr(A) = 0, we have

$$D(A)\Lambda = 0$$

Then

$$D(A)\Lambda^{\otimes q} = [D(A)\Lambda] \otimes \Lambda^{\otimes(q-1)} + \Lambda \otimes [D(A)\Lambda] \otimes \Lambda^{\otimes(q-2)}$$

$$+ \Lambda^{\otimes 2} \otimes [D(A)\Lambda] \otimes \Lambda^{\otimes(q-3)} + \cdots + \Lambda^{\otimes(q-1)} \otimes [D(A)\Lambda] = 0$$

for $q = 1,2,\cdots$. Furthermore

$$D(A)\ (\Lambda^{\otimes q} \otimes T) = [D(A)\Lambda^{\otimes q}] \otimes T + \Lambda^{\otimes q} \otimes [D(A)T]$$

$$= \Lambda^{\otimes q} \otimes [D(A)T]$$

for T in L_t. Since

$$D(A)P = PD(A)$$

for any P in CS_{r+s}, it follows that

$$D(A)[S(1\cdots r)S(r+1\cdots r+s)\ P_{(q,t)}\Lambda^{\otimes q} \otimes T]$$

$$= S(1\cdots r)S(r+1\cdots r+s)\ P_{(q,t)}\Lambda^{\otimes q} \otimes [D(A)T]$$

for T in L_t.

For A,B in su(2), we have

$$\Delta(AB)\ (\Lambda^{\otimes q} \otimes T) = D(A)D(B)(\Lambda^{\otimes q} \otimes T)$$

$$= D(A)\{\Lambda^{\otimes q} \otimes [D(B)T]\} = \Lambda^{\otimes q} \otimes [D(A)D(B)T]$$

$$= \Lambda^{\otimes q} \otimes [\Delta(AB)T]$$

and it follows that

$$\Delta(F)(\Lambda^{\otimes q} \otimes T) = \Lambda^{\otimes q} \otimes [\Delta(F)T]$$

for arbitrary F in $[su(2)]^2$. Since

$$\Delta(AB)P = D(A)D(B)P = D(A)PD(B) = PD(A)D(B)$$

$$= P\Delta(AB)$$

it follows that

$$\Delta(F)P = P\Delta(F)$$

for arbitrary F in $[su(2)]^2$. This leads to the result

$$\Delta(F)[\mathcal{S}(1\cdots r)\mathcal{S}(r+1,\cdots,r+s)p_{(q,t)}\Lambda^{\otimes q} \otimes T]$$

$$= \mathcal{S}(1\cdots r)\mathcal{S}(r+1,\cdots,r+s)p_{(q,t)}\Lambda^{\otimes q} \otimes [\Delta(F)T]$$

for T in $\mathcal{L}_t$.

The Map: $T \rightarrow W^t_{r,s}T$ from $\mathcal{L}_t$ to $\mathcal{L}^t_{r,s}$

Define the linear map $W^t_{r,s}$ from $\mathcal{L}_t$ onto $\mathcal{L}^t_{r,s}$ as follows:

$$W^t_{r,s}T = \mathcal{S}(1\cdots r)\mathcal{S}(r+1,\cdots,r+s)p_{(q,t)}\Lambda^{\otimes q} \otimes T$$

Since $\mathcal{L}^t_{r,s}$ has the same dimension as $\mathcal{L}_t$, this map is one-to-one from $\mathcal{L}_t$ onto $\mathcal{L}^t_{r,s}$. Consequently the inverse $(W^t_{r,s})^{-1}$ of $W^t_{r,s}$ exists. The results derived in the above section can be stated in the form

$$\Gamma(M)W^t_{r,s}T = W^t_{r,s}\Gamma(M)T$$

$$D(A)W^t_{r,s}T = W^t_{r,s}D(A)T$$

$$\Delta(F)W^t_{r,s}T = W^t_{r,s}\Delta(F)T$$

Thus the subspaces $\mathcal{L}^t_{r,s}$ for each t are invariant under the operation of $\Gamma(M)$, $D(A)$ and $\Delta(F)$ on $\mathcal{L}_r \otimes \mathcal{L}_s$.

The Representations $\Gamma^t_{r,s}$, $D^t_{r,s}$, $\Delta^t_{r,s}$
on the Representation Space $\mathcal{L}^t_{r,s}$

The representations $\Gamma^t_{r,s}$ of SU(2), $D^t_{r,s}$ of su(2) (or C[su(2)]), and $\Delta^t_{r,s}$ of $[su(2)]^2$ with the representation space $\mathcal{L}^t_{r,s}$ are given by the homomorphisms: $M \to \Gamma(M)$ for all M in $SU(2)$; $A \to D(A)$ for all A in $su(2)$, and $F \to \Delta(F)$ for all F in $[su(2)]^2$ respectively where the operators $\Gamma(M)$, $D(A)$ and $\Delta(F)$ are restricted to operate on the domain $L^t_{r,s}$.

Let $\Gamma^{(t)}(M)$, $D^{(t)}(A)$, and $\Delta^{(t)}(F)$ denote the restriction of the operators $\Gamma(M)$, $D(A)$, and $\Delta(F)$ respectively to the domain L_t. Let $\Gamma^{(t)}_{(r,s)}(M)$, $D^{(t)}_{(r,s)}(A)$, and $\Delta^{(t)}_{(r,s)}(F)$ denote the restriction of the operators $\Gamma(M)$, $D(A)$, and $\Delta(F)$ respectively to the domain $L^t_{r,s}$. Then the equations of the above section can be expressed in the form

$$\Gamma^{(t)}_{(r,s)}(M)W^t_{r,s} = W^t_{r,s}\,\Gamma^{(t)}(M)$$

$$D^{(t)}_{(r,s)}(A)W^t_{r,s} = W^t_{r,s}\,D^{(t)}(A)$$

$$\Delta^{(t)}_{(r,s)}(F)W^t_{r,s} = W^t_{r,s}\,\Delta^{(t)}(F)$$

from which it follows that

$$\Gamma^{(t)}_{(r,s)}(M) = W^t_{r,s}\Gamma^{(t)}(M)(W^t_{r,s})^{-1}$$

$$D^{(t)}_{(r,s)}(A) = W^t_{r,s}D^{(t)}(A)(W^t_{r,s})^{-1}$$

$$\Delta^{(t)}_{(r,s)}(F) = W^t_{r,s}\Delta^{(t)}(F)(W^t_{r,s})^{-1}$$

Thus the representations $\Gamma^t_{r,s}$, $D^t_{r,s}$ and $\Delta^t_{r,s}$ are equivalent to the representations Γ^t, D^t and Δ^t respectively.

Projection Operator onto $L^t_{r,s}$ and the Inner Product Theorem

Define the operator

$$M^t_{r,s} = S(1\cdots r)S(r+1,\cdots,r+s)A(1,r+s-q+1)A(2,r+s-q+2)\cdots A(q,r+s)$$
$$S(q+1\cdots r+s-q)$$

Lemma 1: $M^t_{r,s}M^t_{r,s} = \alpha^t_{r,s}M^t_{r,s}$ where

$$\alpha^t_{r,s} = (r-q)!(s-q)!(r+s-2q)!q!(1+r+s-q)!/(1+r+s-2q)!$$

for $r+s-q > r+1 > r > q+1$.

Proof:

$$M^t_{r,s}M^t_{r,s} = \mathcal{S}(1\cdots r)\mathcal{S}(r+1,\cdots,r+s)A(1,r+s-q+1)\cdots A(q,r+s)$$
$$\mathcal{S}(q+1\cdots r+s-q)\mathcal{S}(1\cdots r)\mathcal{S}(r+1\cdots r+s)A(1,r+s-q+1)\cdots A(q,r+s)$$
$$\mathcal{S}(q+1\cdots r+s-q)$$
$$= \mathcal{S}(1\cdots r)\mathcal{S}(r+1\cdots r+s)\mathcal{S}(q+1\cdots r+s-q)A(q,r+s)\cdots A(1,r+s-q+1)$$
$$\mathcal{S}(1\cdots r)\mathcal{S}(r+1\cdots r+s)A(1,r+s-q+1)\cdots A(q,r+s)\mathcal{S}(q+1\cdots r+s-q)$$

since $A(i,r+s-q+i)$ commutes with $A(j,r+s-q+j)$ for $i \neq j$ and $1 \leq i,\ j \leq q$. Also $\mathcal{S}(q+1\cdots r+s-q)$ commutes with $A(i,r+s-q+i)$ for $1 \leq i \leq q$.

We have

$$A(q,r+s)\cdots A(1,r+s-q+1)\mathcal{S}(1\cdots r)\mathcal{S}(r+1\cdots r+s)A(1,r+s-q+1)\cdots A(q,r+s)$$
$$= A(q,r+s)\cdots A(1,r+s-q+1)[1+(12)+(13)+\cdots+(1r)]$$
$$[1+(r+1,r+s-q+1)+\cdots+(r+s-q,r+s-q+1)+(r+s-q+2,r+s-q+1)+\cdots+$$
$$(r+s,r+s-q+1)]A(1,r+s-q+1)\mathcal{S}(2\cdots r)\mathcal{S}(r+1\cdots r+s-q,r+s-q+2\cdots r+s)$$
$$A(2,r+s-q+2)\cdots A(q,r+s) \tag{A_1}$$

since

$$\mathcal{S}(1\cdots r) = [1+(12)+(13)+\cdots+(1r)]\mathcal{S}(2\cdots r)$$

and

$$\mathcal{S}(r+1\cdots r+s) = [1+(r+1,r+s-q+1) + \cdots +(r+s-q,r+s-q+1)$$
$$+ (r+s-q+2,r+s-q+1)+\cdots+(r+s,r+s-q+1)]\mathcal{S}(r+1\cdots r+s-q,r+s-q+2\cdots r+s)$$

Now

$$A(1,r+s-q+1)[1+(12)+\cdots+(1r)][1+(r+1,r+s-q+1)+\cdots+$$

$$(r+s,r+s-q+1)]A(1,r+s-q+1)$$

$$= [(r-1)+(s-1)+2+\sum_{\substack{j=r+1\\ j\neq r+s-q+1}}^{r+s}\ \sum_{i=2}^{r} A(i,j)]\ A(1,r+s-q+1)$$

$$= [(r-1)+(s-1)+2+(r-1)(s-1)-\sum_{\substack{j=r+1\\ j\neq r+s-q+1}}^{r+s}\ \sum_{i=2}^{r}(i,j)]A(1,r+s-q+1)$$

by Theorem 1 and Corollaries 1 and 2 in the paragraph "Antisymmetrization and Symmetrization Operators in CS_r."

Next

$$A(2,r+s-q+2)[(r-1)+(s-1)+2+(r-1)(s-1)-\sum_{\substack{j=r+1\\ j\neq r+s-q+1}}^{r+s}\ \sum_{i=2}^{r}(i,j)]$$

$$= [(r-1)+(s-1)+2+(r-1)(s-1)+1-\sum_{\substack{j=r+1\\ j\neq r+s-q+1\\ j\neq r+s-q+2}}^{r+s}\ \sum_{i=3}^{r}(i,j)]A(2,r+s-q+2) \qquad (B_1)$$

when acting on the remaining factors, since

$$A(2,r+s-q+2)(2,r+s-q+2) = -A(2,r+s-q+2)$$

$$A(2,r+s-q+2)(2,j) = (2,j)A(j,r+s-q+2)$$

$$A(2,r+s-q+2)(i,r+s-q+2) = (i,r+s-q+2)A(2,i)$$

$$A(2,r+s-q+2)(i,j) = (i,j)A(2,r+s-q+2)$$

$$A(j,r+s-q+2)S(r+1\cdots r+s-q,r+s-q+2\cdots r+s) = 0$$

$$A(2,i)S(2\cdots r) = 0$$

for $j=r+1,\cdots,r+s-q,r+s-q+3,\cdots,r+s$ and $i=3,4,\cdots,r$.

Repeating this procedure acting with the q-2 factors $A(3,r+s-q+3),\cdots,A(q,r+s)$ consecutively, we have

$$[(r-1)+(s-1)+2+(r-1)(s-1)+(q-1)-\sum_{j=r+1}^{r+s-q}\sum_{i=q+1}^{r}(i,j)]A(q,r+s)$$

$$\cdots A(1,r+s-q+1)$$

Bring $A(1,r+s-q+1)$ to the last position of $M^t_{r,s}M^t_{r,s}$, then consider

$$A(2,r+s-q+2)S(2\cdots r)S(r+1\cdots r+s-q,r+s-q+2\cdots r+s)A(2,r+s-q+2)$$

Repeating (A_1) procedure, we have

$$[(r-2)+(s-2)+2+(r-2)(s-2)-\sum_{\substack{j=r+1\\ j\neq r+s-q+1\\ j\neq r+s-q+2}}^{r}\sum_{i=3}^{r}(i,j)]A(2,r+s-q+2)$$

$$\cdot S(3\cdots r)S(r+1\cdots r+s-q,r+s-q+3\cdots r+s)$$

Repeating (B_1) procedure acting with $A(3,r+s-q+3),\cdots,A(q,r+s)$ consecutively, we have

$$[(r-2)+(s-2)+2+(r-2)(s-2)+(q-2)-\sum_{j=r+1}^{r+s-q}\sum_{i=q+1}^{r}(i,j)]$$

$$\cdot\ A(q,r+s)\cdots A(3,r+s-q+3)A(2,r+s-q+2)$$

Bring $A(2,r+s-q+2)$ to the last position of $M^t_{r,s}M^t_{r,s}$, then consider

$$A(3,r+s-q+3)S(3\cdots r)S(r+1\cdots r+s-q,r+s-q+3\cdots r+s)A(3,r+s-q+3)$$

From the above derivation, we knock off one element from $S(1\cdots r)$ and $S(r+1\cdots r+s)$ respectively each time, and derive a factor

$$[(r-k)+(s-k)+2+(r-k)(s-k)+(q-k)-\sum_{j=r+1}^{r+s-q}\sum_{i=q+1}^{r}(i,j)]$$

Finally this procedure ends after the q-th step. Hence the remaining parts are $S(q+1\cdots r)$,$S(r+1\cdots r+s-q)$ and the final factors are

$$\prod_{k=1}^{q}[r-k)+(s-k)+2+(r-k)(s-k)+(q-k)-\sum_{j=r=1}^{r+s-q}\sum_{i=q+1}^{r}(i,j)]$$

Thus

$$M^t_{r,s}M^t_{r,s} = \mathcal{S}(1\cdots r)\mathcal{S}(r+1\cdots r+s)\mathcal{S}(q+1\cdots r+s-q)$$

$$\prod_{k=1}^{q}\ [(r-k)+(s-k)+2+(r-k)(s-k)+(q-k) - \sum_{j=r+1}^{r+s-q}\ \sum_{i=q+1}^{r}(i,j)]\mathcal{S}(q+1\cdots r)$$

$$\cdot\mathcal{S}(r+1)\cdots r+s-q)\mathcal{S}(q+1\cdots r+s-q)A(1,r+s-q+1)\cdots A(q,r+s)$$

$$= (r-q)!(s-q)!\mathcal{S}(1\cdots r)\mathcal{S}(r+1\cdots r+s)\mathcal{S}(q+1\cdots r+s-q)$$

$$\prod_{k=1}^{q}\ [(r-k)+(s-k)+2+(r-k)(s-k)+(q-k)- \sum_{j=r+1}^{r+s-q}\ \sum_{i=q+1}^{r}(i,j)]\mathcal{S}(q+1\cdots r+s-q)$$

$$\cdot A(1,r+s-q+1)\cdots A(q,r+s)$$

for

$$\mathcal{S}(q+1\cdots r)\mathcal{S}(q+1\cdots r+s-q) = (r-q)!\mathcal{S}(q+1\cdots r+s-q)$$

$$\mathcal{S}(r+1\cdots r+s-q)\mathcal{S}(q+1\cdots r+s-q) = (s-q)!\mathcal{S}(q+1\cdots r+s-q)$$

Thus

$$M^t_{r,s}M^t_{r,s} = (r-q)!(s-q)!\mathcal{S}(1\cdots r)\mathcal{S}(r+1\cdots r+s)\mathcal{S}(q+1\cdots r+s-q)$$

$$\prod_{k=1}^{q}\ [(r-k)+(s-k)+2+(r-k)(s-k)+(q-k)-(r-q)(s-q)$$

$$+\sum_{j=r+1}^{r+s-q}\ \sum_{i=q+1}^{r}\ A(i,j)]\mathcal{S}(q+1\cdots r+s-q)A(1,r+s-q+1)\cdots A(q,r+s)$$

$$= (r-q)!(s-q)!\mathcal{S}(1\cdots r)\mathcal{S}(r+1\cdots r+s)\mathcal{S}(q+1\cdots r+s-q)$$

$$\cdot\prod_{k=1}^{q}[(r-k)+(s-k)+2+(r-k)(s-k)+(q-k)-(r-q)(s-q)]\mathcal{S}(q+1\cdots r+s-q)$$

$$\cdot A(1,r+s-q+1)\cdots A(q,r+s)$$

since

$$\mathcal{S}(q+1\cdots r+s-q)A(i,j) = 0$$

for $i = q+1,\cdots,r, j = r+1,\cdots,r+s-q$.

Hence

$$M^t_{r,s}M^t_{r,s} = (r-q)!(s-q)! \prod_{k=1}^{q}[(r-k)+(s-k)+2+(r-k)(s-k)+(q-k)$$

$$-(r-q)(s-q)]\mathcal{S}(1\cdots r)\mathcal{S}(r+1\cdots r+s-q)\mathcal{S}(q+1\cdots r+s-q)\mathcal{S}(q+1\cdots r+s-q)$$

$$\cdot A(1,r+s-q+1)\cdots A(q,r+s)$$

$$= (r-q)!(s-q)!(r+s-2q)! \prod_{k=1}^{q}[(r-k)+(s-k)+2+(r-k)(s-k)$$

$$+ (q-k)-(r-q)(s-q)]M^t_{r,s}$$

$$= [(r-q)!(s-q)!(r+s-2q)!q!(1+r+s-q)!/(1+r+s-2q)!]M^t_{r,s} \qquad \text{Q.E.D.}$$

The transpose $\tilde{M}^t_{r,s}$ of $M^t_{r,s}$ is

$$\tilde{M}^t_{r,s} = \mathcal{S}(q+1\cdots r+s-q)A(q,r+s)\cdots A(1,r+s-q+1)\mathcal{S}(r+1\cdots r+s)\mathcal{S}(1\cdots r)$$

Lemma 2: $\tilde{M}^t_{r,s}M^t_{r,s} = \beta^t_{r,s}\ \mathcal{S}(q+1\cdots r+s-q)A(1,r+s-q+1)\cdots A(q,r+s)$

where $\beta^t_{r,s} = r!s!\alpha^t_{r,s}$.

Proof: $\tilde{M}^t_{r,s}M^t_{r,s} = \mathcal{S}(q+1\cdots r+s-q)A(q,r+s)\cdots A(1,r+s-q+1)$

$\mathcal{S}(r+1\cdots r+s)\mathcal{S}(1\cdots r)\mathcal{S}(1\cdots r)\mathcal{S}(r+1\cdots r+s)A(1,r+s-q+1)\cdots$

$A(q,r+s)\mathcal{S}(q+1\cdots r+s-q)$

$= r!s!\mathcal{S}(q+1\cdots r+s-q)[A(q,r+s)\cdots A(1,r+s-q+1)\mathcal{S}(1\cdots r)$

$\mathcal{S}(r+1\cdots r+s)A(1,r+s-q+1)\cdots A(q,r+s)]\mathcal{S}(q+1\cdots r+s-q)$

By Lemma 1 above, the quantity in the parenthesis $[\cdots]$ can be replaced by $[(r-q)!(s-q)!q!(1+r+s-q)!/(1+r+s-2q)!]A(1,r+s-q+1)\cdots A(q,r+s)$.

Thus

$$\tilde{M}^{t}_{r,s}M^{t}_{r,s} = [r!s!(r-q)!(s-q)!q!(1+r+s-q)!/(1+r+s-2q)!]$$

$$\cdot\mathcal{S}(q+1\cdots r+s-q)A(1,r+s-q+1)\cdots A(q,r+s)\mathcal{S}(q+1\cdots r+s-q)$$

$$= [r!s!(r-q)!(s-q)!(r+s-2q)!q!(1+r+s-q)!/(1+r+s-2q)!]$$

$$\cdot\mathcal{S}(q+1\cdots r+s-q)A(1,r+s-q+1)\cdots A(q,r+s)$$

i.e.

$$\beta^{t}_{r,s} = r!s!\alpha^{t}_{r,s} \qquad \text{Q.E.D.}$$

Lemma 3:

(1) For p,q in S_{r+s}

$$pq = p\begin{Bmatrix} 1 & 2 & 3\cdots & r+s \\ q1 & q2 & q3\cdots & q(r+s) \end{Bmatrix} = \begin{Bmatrix} p1 & p2 & p3 & \cdots & p(r+s) \\ pq1 & pq2 & pq3 & \cdots & pq(r+s) \end{Bmatrix} p$$

(2) $p_{(q,t)}(2i-1,2i) = (i,t+q+i)p_{(q,t)}$ for $i=1,2,\cdots,q$.

(3) $p_{(q,t)}\mathcal{S}(2q+1\cdots 2q+t) = \mathcal{S}(q+1\cdots q+t)p_{(q,t)}$

Proof: (1) and (2) can be derived by the definitions. For (3): Let R be the set of permutations in S_{r+s} which permutes elements of $\{2q+1,2q+2,\cdots,2q+t\}$ among themselves and acts as the identity outside this set. Then

$$p_{(q,t)}\ \mathcal{S}(2q+1\cdots 2q+t)$$

$$= \sum_{p\in R} (p_{(q,t)}p)$$

$$= \sum_{p\in R} \begin{Bmatrix} p_{(q,t)}(2q+1) & p_{(q,t)}(2q+2)\cdots & p_{(q,t)}(2q+t) \\ p_{(q,t)}p(2q+1) & p_{(q,t)}p(2q+2)\cdots & p_{(q,t)}p(2q+t) \end{Bmatrix} p_{(q,t)}$$

by (2). Then

$$p_{(q,t)}S(2q+1\cdots 2q+t)$$

$$= \sum_{p\in R}\left\{\overset{q+1}{p_{(q,t)}p(2q+1)}\ \overset{q+2}{p_{(q,t)}p(2q+2)}\cdots\overset{q+t}{p_{(q,t)}p(2q+t)}\right\} p_{(q,t)}$$

$$= S(q+1\cdots q+t)\ p_{(q,t)}$$

since each $p_{(q,t)}p$ maps $\{2q+1,2q+2,\cdots,2q+t\}$ onto $\{q+1,q+2,\cdots,q+t\}$. Q.E.D.

Lemma 4: $W^t_{r,s}T = (\frac{1}{2})^q \frac{1}{t!} M^t_{r,s}\ p_{(q,t)}\Lambda^{\otimes q} \otimes T$

Proof:

$$W^t_{r,s}T = S(1\cdots r)S(r+1\cdots r+s)p_{(q,t)}\Lambda^{\otimes q} \otimes T$$

$$= S(1\cdots r)S(r+1\cdots r+s)p_{(q,t)}\ [(\tfrac{1}{2}\,A(1,2)\Lambda)^{\otimes q} \otimes \tfrac{1}{t!}\,S(1\cdots t)T]$$

since $\frac{1}{2}A(1,2)\Lambda = \Lambda$ and $T = \frac{1}{t!}S(1\cdots t)T$ iff $T \in L_t$.

Thus

$$W^t_{r,s}T = S(1\cdots r)S(r+1\cdots r+s)p_{(q,t)}\ [(\tfrac{1}{2})^q A(1,2)A(3,4)\cdots$$

$$A(2q-1,2q)\Lambda^{\otimes q} \otimes \tfrac{1}{t!}\,S(1\cdots t)T]$$

$$= S(1\cdots r)S(r+1\cdots r+s)(\tfrac{1}{2})^q\ \tfrac{1}{t!}\ p_{(q,t)}\ [A(1,2)A(3,4)\cdots$$

$$A(2q-1,2q)S(2q+1\cdots 2q+t)\Lambda^{\otimes q} \otimes T]$$

$$= (\tfrac{1}{2})^q\ \tfrac{1}{t!}\,S(1\cdots r)S(r+1\cdots r+s)A(1,r+s-q+1)\cdots$$

$$A(q,r+s)S(q+1\cdots r+s-q)p_{(q,t)}\Lambda^{\otimes q} \otimes T$$

by Lemma 3 above. Thus

$$W^t_{r,s}T = (\tfrac{1}{2})^q\ \tfrac{1}{t!}\ M^t_{r,s}p_{(q,t)}\Lambda^{\otimes q} \otimes T \qquad \text{Q.E.D.}$$

Lemma 5: $\tilde{p}_{(q,t)} = \hat{p}_{(q,t)}$ where $\tilde{p}_{(q,t)}$ and $\hat{p}_{(q,t)}$ are the transpose and the inverse of $p_{(q,t)}$ respectively.

Proof: $(\tilde{p}_{(q,t)}\bar{u}_1\bar{u}_2\cdots\bar{u}_{2q}\bar{u}_{2q+1}\cdots\bar{u}_{2q+t})\circ(v_1v_2\cdots v_{2q+t})$

$$= (\bar{u}_1\bar{u}_2\cdots\bar{u}_{2q}\bar{u}_{2q+1}\cdots\bar{u}_{2q+t})\circ(p_{(q,t)}v_1v_2\cdots v_{2q}v_{2q+1}\cdots v_{2q+t})$$

$$= (\bar{u}_1\bar{u}_2\cdots\bar{u}_{2q}\bar{u}_{2q+1}\cdots\bar{u}_{2q+t})\circ(v_1v_3\cdots v_{2q-1}v_{2q+1}\cdots v_{2q+t}v_2v_4\cdots v_{2q})$$

$$= (\bar{u}_1\circ v_1)(\bar{u}_2\circ v_3)(\bar{u}_3\circ v_5)\cdots(\bar{u}_q\circ v_{2q-1})(\bar{u}_{q+1}\circ v_{2q+1})\cdots(\bar{u}_{q+t}\circ v_{2q+t})$$
$$(\bar{u}_{q+t+1}\circ v_2)\cdots(\bar{u}_{2q+t}\circ v_{2q})$$

$$= (\bar{u}_1\circ v_1)(\bar{u}_{q+t+1}\circ v_2)(\bar{u}_2\circ v_3)(\bar{u}_{q+t+2}\circ v_4)\cdots(\bar{u}_{2q+t}\circ v_{2q})\cdots$$
$$(\bar{u}_{q+t}\circ v_{2q+t})$$

$$= (\bar{u}_1\bar{u}_{q+t+1}\bar{u}_2\bar{u}_{q+t+2}\cdots\bar{u}_q\bar{u}_{2q+t}\bar{u}_{q+1}\cdots\bar{u}_{q+t})\circ(v_1v_2\cdots v_{2q+t})$$

$$= (\hat{p}_{(q,t)}\bar{u}_1\bar{u}_2\cdots\bar{u}_{2q+t})\circ(v_1v_2\cdots v_{2q+t})$$

for all $\bar{u}_i$'s in $\bar{U}_2$ and v_i's in U_2. Thus

$$\tilde{p}_{(q,t)} = \hat{p}_{(q,t)}$$

Q.E.D.

Inner Product Theorem: Given T and S in L_t and the associated $W^t_{r,s}T$ and $W^t_{r,s}S$ in $L^t_{r,s}$, we have

$$<W^t_{r,s}T \mid W^t_{r,s}S> \;=\; A_{rst} <T \mid S> .$$

where

$$A_{rst} = r!s!(r-q)!(s-q)!q!(1+r+s-q)!/(1+r+s-2q)!$$

Proof:

$$<W^t_{r,s}T \mid W^t_{r,s}S>$$

$$= <(\tfrac{1}{2})^q \tfrac{1}{t!} M^t_{r,s}P_{(q,t)}\Lambda^{\otimes q} \otimes T \mid (\tfrac{1}{2})^q \tfrac{1}{t!} M^t_{r,s}P_{(q,t)}\Lambda^{\otimes q} \otimes S>$$

$$= [(\tfrac{1}{2})^q \tfrac{1}{t!}]^2 <M^t_{r,s}P_{(q,t)}\Lambda^{\otimes q} \otimes T \mid M^t_{r,s}P_{(q,t)}\Lambda^{\otimes q} \otimes S>$$

$$= [(\tfrac{1}{2})^q \tfrac{1}{t!}]^2 <\tilde{M}^t_{r,s}M^t_{r,s}P_{(q,t)}\Lambda^{\otimes q} \otimes T \mid P_{(q,t)}\Lambda^{\otimes q} \otimes S>$$

$$= [(\tfrac{1}{2})^q \tfrac{1}{t!}]^2 \beta^t_{r,s} <\mathcal{S}(q+1\cdots r+s-q)A(1,r+s-q+1)A(2,r+s-q+2)$$

$$\cdots A(q,r+s)P_{(q,t)}\Lambda^{\otimes q} \otimes T \mid P_{(q,t)}\Lambda^{\otimes q} \otimes S>$$

$$= (\tfrac{1}{2})^q \tfrac{1}{t!} \beta^t_{r,s} <P_{(q,t)}\Lambda^{\otimes q} \otimes T \mid P_{(q,t)}\Lambda^{\otimes q} \otimes S>$$

$$= (\tfrac{1}{2})^q \tfrac{1}{t!} \beta^t_{r,s} <\tilde{P}_{(q,t)}P_{(q,t)}\Lambda^{\otimes q} \otimes T \mid {}^{\otimes q} \otimes S>$$

$$= (\tfrac{1}{2})^q \tfrac{1}{t!} \beta^t_{r,s}<\Lambda^{\otimes q} \otimes T \mid \Lambda^{\otimes q} \otimes S>$$

since

$$\mathcal{S}(q+1\cdots r+s-q)A(1,r+s-q+1)A(2,r+s-q+2)\cdots A(q,r+s)P_{(q,t)}\Lambda^{\otimes q} \otimes T$$

$$= (t!2^q)P_{(q,t)}\Lambda^{\otimes q} \otimes T$$

Thus

$$<W^t_{r,s}T \mid W^t_{r,s}S>$$

$$= (\tfrac{1}{2})^q \tfrac{1}{t!} \beta^t_{r,s} \; 2^q <T \mid S>$$

$$= \tfrac{1}{t!} \beta^t_{r,s} <T \mid S>$$

$$= A_{rst} <T \mid S>$$

since

$$\begin{aligned}
&< \Lambda^{\otimes q} \otimes T | \Lambda^{\otimes q} \otimes S> \\
&= \overline{(\Lambda^{\otimes q} \otimes T)} \odot (\Lambda^{\otimes q} \otimes S) \\
&= (\overline{\Lambda^{\otimes q}} \otimes \overline{T}) \odot (\Lambda^{\otimes q} \otimes S) \\
&= (\overline{\Lambda^{\otimes q}} \odot \Lambda^{\otimes q})(\overline{T} \odot S) \\
&= 2^q \, (\overline{T} \odot S) \\
&= 2^q \, <T \,|\, S>
\end{aligned}$$

Q.E.D.

Corollary: $<W^t_{r,s} T \,|\, W^t_{r,s} S> \; = rsq(1+r+s-q) \; <W^t_{r-1,s-1} T \,|\, W^t_{r-1,s-1} S>$.

Proof: By the Inner Product Theorem with r,s,q replaced respectively by r-1,s-1,q-1 we get

$$\begin{aligned}
&<W^t_{r-1,s-1} T \,|\, W^t_{r-1,s-1} S> \\
&= (r-1)!(s-1)!(r-q)!(s-q)!(q-1)!(r+s-q)!/(1+r+s-2q)! <T|S>
\end{aligned}$$

The result follows. Q.E.D.

Note that $\frac{1}{\alpha^t_{r,s}} M^t_{r,s}$ is the projection operator on $L^t_{r,s}$. By Lemma 1 and 4 we have

$$\frac{1}{\alpha^t_{r,s}} M^t_{r,s} \, (W^t_{r,s} T) = W^t_{r,s} T$$

for $T \, \varepsilon L_t$.

We can also prove that $L^t_{r,s}$ has the same dimension as L_t, which is t+1. Take any orthonormal basis $T_1, T_2, \cdots, T_{t+1}$ of L_t. Then $(A_{rst})^{-\frac{1}{2}} W^t_{r,s} T_k$ with k=1,2,$\cdots$,t+1 is an orthonormal set which spans

$L^t_{r,s}$. The orthonormality implies linear independence, consequently this set is a basis for $L^t_{r,s}$. Therefore the dimension of $L^t_{r,s}$ is t+1.

We state here a convenient form of the result of the Theorem with $r=2j_1, s=2j_2, t=2j, q=j_1+j_2-j$ and $B_{j_1j_2j} = A_{2j_1,2j_2,2j}$. For T,S in L_{2j} we have

$$\langle W^{2j}_{2j_1,2j_2} T \mid W^{2j}_{2j_1,2j_2} S\rangle = B_{j_1j_2j} \langle T \mid S\rangle$$

where

$$B_{j_1j_2j} = \frac{(2j_1)!(2j_2)!(j_1+j_2-j)!(j_1-j_2+j)!(j_2-j_1+j)!(1+j_1+j_2+j)!}{(1+2j)!}$$

Another Orthonormal Basis for $L_{2j_1} \otimes L_{2j_2}$

We shall construct a basis for $L_{2j_1} \otimes L_{2j_2}$ which are simultaneous eigentensors of a given set of operators to be prescribed. Define

$$\Phi_{j_1j_2}(j,m) = (B_{j_1j_2j})^{-\frac{1}{2}} W^{2j}_{2j_1,2j_2} \Phi(j,m)$$

$$= (B_{j_1j_2j})^{-\frac{1}{2}} S(1\cdots 2j_1)S(2j_1+1\cdots 2j_1+2j_2)P_{j_1j_2j}$$

$$[\Lambda^{\otimes(j_1+j_2-j)} \otimes \Phi(j,m)]$$

where

$$P_{j_1j_2j} = P_{(j_1+j_2-j,2j)}$$

The allowed values for j_1,j_2,j,m are seen to be $j_1=0, \frac{1}{2}, 1,\cdots;$ $j_2=0, \frac{1}{2}, 1,\cdots;$ $j=|j_1-j_2|,|j_1-j_2|+1,|j_1-j_2|+2,\cdots,j_1+j_2;$ $m=-j,-j+1,-j+2,\cdots,j.$

For fixed j_1, j_2, j, it follows from the Theorem that $\Phi_{j_1 j_2}(j,m)$ for the allowed values of m is an orthonormal basis of $L^{2j}_{2j_1,2j_2}$. It is also true that $\Phi_{j_1 j_2}(j,m)$ and $\Phi_{j_1 j_2}(j',m')$ are orthogonal for $j \neq j'$ since they are eigentensors of the Hermitian operator $\Delta(J^2)$ corresponding to different eigenvalues (the eigenvalues will soon be given). Thus the subspaces $L^{2j}_{2j_1,2j_2}$ for fixed j_1, j_2 and variable j are orthogonal to one another. Since $L_{2j_1} \otimes L_{2j_j}$ is the direct sum of the subspaces $L^{2j}_{2j_1,2j_2}$, i.e.

$$L_{2j_1} \otimes L_{2j_2} = \bigoplus_{\substack{j=|j_1-j_2| \\ \Delta j=1}}^{j_1+j_2} L^{2j}_{2j_1,2j_2}$$

we see that $\Phi_{j_1 j_2}(j,m)$ for fixed j_1, j_2 and all allowed values of j,m form an orthonormal basis of $L_{2j_1} \otimes L_{2j_2}$.

Since all tensors in $L_{2j_1} \otimes L_{2j_2}$ are eigentensors of $\Delta_1(J^2)$ and $\Delta_2(J^2)$ with the corresponding eigenvalues $j_1(j_1+1)$ and $j_2(j_2+1)$ respectively, we have

$$\Delta_1(J^2)\Phi_{j_1 j_2}(j,m) = j_1(j_1+1)\,\Phi_{j_1 j_2}(j,m)$$

$$\Delta_2(J^2)\Phi_{j_1 j_2}(j,m) = j_2(j_2+1)\,\Phi_{j_1 j_2}(j,m)$$

We also have

$$\Delta(J^2)\Phi_{j_1,j_2}(j,m) = (B_{j_1 j_2 j})^{-\frac{1}{2}}\, W^{2j}_{2j_1,2j_2}[\Delta(J^2)\Phi(j,m)]$$

$$= j(j+1)\,\Phi_{j_1 j_2}(j,m)$$

$$D(J_z)\Phi_{j_1j_2}(j,m) = (B_{j_1j_2j})^{-\frac{1}{2}} W^{2j}_{2j_1,2j_2}\ [D(J_z)\Phi(j,m)]$$

$$= m\ \Phi_{j_1j_2}(j,m)$$

Thus $\Phi_{j_1j_2}(j,m)$ is a simultaneous eigentensor of $\Delta_1(J^2)$ with eigenvalue $j_1(j_1+1)$, of $\Delta_2(J^2)$ with eigenvalue $j_2(j_2+1)$, of $\Delta(J^2)$ with eigenvalue $j(j+1)$ and of $D(J_z)$ with eigenvalue m.

Clebsch-Gordan Coefficients $C_{j_1j_2}(m_1,m_2;j,m)$

We have the two bases $\Phi(j_1,m_1;j_2,m_2)$ (all allowed m_1,m_2) and $\Phi_{j_1j_2}(j,m)$ (all allowed j,m) of $L_{2j_1} \otimes L_{2j_2}$ for fixed j_1,j_2. The expression of one basis in terms of the other basis defines the Clebsch-Gordan coefficients $C_{j_1j_2}(m_1,m_2;j,m)$, i.e.

$$\Phi_{j_1j_2}(j,m) = \sum_{m_1,m_2} C_{j_1j_2}(m_1,m_2;j,m)\ \Phi(j_1,m_1;j_2,m_2)$$

where the sum is over all allowed values of m_1,m_2. It follows from the orthonormality of $\Phi(j_1,m_1;j_2,m_2)$ (all allowed m_1,m_2) that

$$C_{j_1j_2}(m_1,m_2;j,m) = <\Phi(j_1,m_1;j_2,m_2\ |\Phi_{j_1j_2}(j,m)>$$

Derivation of an Expression for the Clebsch-Gordan Coefficients

Now we shall give a derivation of an expression for the Clebsch-Gordan coefficients. First we need the following equation which is obtained from the expression for $G(\alpha,\beta)$ with α=j+m and β=j-m.

$$\Phi(j,m) = [(j+m)!(j-m)!]^{\frac{1}{2}}\ [(2j)!]^{-\frac{1}{2}}\ G(j+m,j-m)$$

$$= [(2j)!(j+m)!(j-m)!]^{-\frac{1}{2}}\ S(1\cdots 2j)[g_1^{\otimes(j+m)} \otimes g_2^{\otimes(j-m)}]$$

Substituting the expression for $\Phi(j,m)$ into the equation

$$\Phi_{j_1j_2}(j,m) = (B_{j_1j_2j})^{-\frac{1}{2}} \mathcal{S}(1\cdots 2j_1)\mathcal{S}(2j_1+1\cdots 2j_1+2j_2)p_{j_1j_2j}$$
$$[\Lambda^{\otimes(j_1+j_2-j)} \otimes \Phi(j,m)]$$

gives

$$\Phi_{j_1j_2}(j,m) = (B_{j_1j_2j})^{-\frac{1}{2}} \mathcal{S}(1\cdots 2j_1)\mathcal{S}(2j_1+1\cdots 2j_1+2j_2)p_{j_1j_2j}$$
$$\Lambda^{\otimes(j_1+j_2-j)} \otimes \{[(2j)!]^{-\frac{1}{2}}(j_1-j_2+j)!(j_2-j_1+j)!\ [(j+m)!(j-m)!]^{\frac{1}{2}}$$
$$\sum_k [(j_1-j_2+j-k)!(j+m-k)!(j_2-j_1-m+k)!]^{-1}\ [g_1^{\otimes k} \otimes g_2^{\otimes(j_1-j_2+j-k}]$$
$$\otimes [g_1^{\otimes(j+m-k)} \otimes g_2^{\otimes(j_2-j_1-m+k)}]\}$$

where k is summed over all values which give rise to no factorials of negative integers in the summand. This result was obtained by collecting together all the monomial terms of $\Phi(j,m)$ in which g_1 appears k times in the first j_1-j_2+j places, and then summing over k. Because of the operator $\mathcal{S}(1\cdots 2j_1)\mathcal{S}(2j_1+1\cdots 2j_1+2j_2)p_{j_1j_2j}$, any permutation of a monomial term of $\Phi(j,m)$ which permutes the first j_1-j_2+j vectors among themselves or permutes the last j_2-j_1+j vectors among themselves does not change the contribution from that monomial term of $\Phi(j,m)$ to the result. Thus, each monomial term in which g_1 appears k times in the first j_1-j_2+j places (and consequently $j+m-k$ times in the last j_2-j_1+j places) can be replaced by

$$[g_1^{\otimes k} \otimes g_2^{\otimes(j_1-j_2+j-k)}] \otimes [g_1^{\otimes(j+m-k)} \otimes g_2^{\otimes(j_2-j_1-m+k)}]$$

which is such a permutation of it. The factor $(j+m)!(j-m)!$ appears because that is the number of times each distinct monomial appears in

$$S(1\cdots 2j)[g_1^{\otimes(j+m)} \otimes g_2^{\otimes(j-m)}]$$

which equals the number of permutations of j+m objects times the number of permutations of j-m objects. The factor

$$\frac{(j_1-j_2+j)!}{k!(j_1-j_2+j-k)!}\ \frac{(j_2-j_1+j)!}{(j+m-k)!(j_2-j_1-m+k)!}$$

$$= (j_1-j_2+j)!(j_2-j_1+j)![(j_1-j_2+j-k)!(j+m-k)!k!(j_2-j_1-m+k)!]^{-1}$$

is the number of distinct monomials in which g_1 and g_2 appear k times and j_1-j_2+j-k times respectively in the first j_1-j_2+j places while g_1 and g_2 appear j+m-k times and j_2-j_1-m+k times respectively in the last j_2-j_1+j places, which equals the number of combinations of j_1-j_2+j objects taken k at a time multiplied by the number of combinations of j_2-j_1+j objects taken j+m-k times.

Now, the substitution of the following

$$\Lambda^{\otimes(j_1+j_2-j)} = (g_1g_2-g_2g_1)^{\otimes(j_1+j_2-j)}$$

into the original expression for $\Phi_{j_1j_2}(j,m)$ gives

$$\Phi_{j_1j_2}(j,m) = (B_{j_1j_2j})^{-\frac{1}{2}}S(1\cdots 2j_1)S(2j_1+1\cdots 2j_1+2j_2)P_{j_1j_2j}$$

$$\{(j_1+j_2-j)!\sum_s(-1)^s[s!(j_1+j_2-j-s)!]^{-1}[g_1g_2)^{\otimes(j_1+j_2-j-s)}$$

$$\otimes (g_2g_1)^{\otimes s}\} \otimes \Phi(j,m)$$

where s is summed over all values which give rise to no factorials of negative integers in the summand. This was obtained by taking the binomial expansion of $(g_1g_2-g_2g_1)^{\otimes(j_1+j_2-j)}$, but the following fact was needed. Due to the presence of the operation

$$S(1\cdots 2j_1)S(2j_1+1\cdots 2j_1+2j_2)p_{j_1j_2j}$$

any permutation of a monomial term of $\Lambda^{\otimes(j_1+j_2-j)}$ which permutes the vectors in the odd numbered positions among themselves or the vectors in the even numbered positions among themselves will not change the contribution of that monomial term the result. Now when $\Lambda^{\otimes(j_1+j_2-j)}$ is expressed as a polynomial in the two quantities g_1g_2 and g_2g_1, all permutations of these pairs among themselves are valid. This is true because such a permutation only permutes vectors in odd numbered positions among themselves while permuting vectors in even numbered positions among themselves. The factor

$$(j_1+j_2-j)!(-1)^s[s!(j_1+j_2-j-s)!]^{-1}$$

appears because it is the binomial coefficient of $x^{j_1+j_2-j-s}y^s$ in the expansion of $(x-y)^{j_1+j_2-j}$.

We now have two expressions for $\Phi_{j_1j_2}(j,m)$, one where $\Phi(j,m)$ is replaced by a sum and one where $\Lambda^{\otimes(j_1+j_2-j)}$ is replaced by a sum. Combining these two results and carrying out the operation $p_{j_1j_2j}$ gives

$$\Phi_{j_1j_2}(j,m) = K_{j_1j_2jm}S(1\cdots 2j_1)S(2j_1+1\cdots 2j_1+2j_2)\sum_s\sum_k$$

$$(-1)^s[s!(j_1+j_2-j-s)!(j_1-j_2+j-k)!(j+m-k)!k!(j_2-j_1-m+k)!]^{-1}$$

$$[g_1^{\otimes(j_1+j_2-j-s)}\otimes g_2^{\otimes s}\otimes g_1^{\otimes k}\otimes g_2^{\otimes(j_1-j_2+j-k)}]\otimes[g_1^{\otimes(j+m-k)}$$

$$\otimes g_2^{\otimes(j_2-j_1-m+k)}\otimes g_2^{\otimes(j_1+j_2-j-s)}\otimes g_1^{\otimes s}]$$

$$= K_{j_1 j_2 jm}\, S(1\cdots 2j_1) S(2j_1+1\cdots 2j_1+2j_2) \sum_s \sum_k (-1)^s$$
$$[s!(j_1+j_2-j-s)!(j_1-j_2+j-k)!(j+m-k)!k!(j_2-j_1-m+k)!]^{-1}$$
$$[g_1^{\otimes(j_1+j_2-j-s+k)} \otimes g_2^{\otimes(j_1-j_2+j+s-k)}] \otimes [g_1^{\otimes(j+m+s-k)}$$
$$\otimes g_2^{\otimes(2j_2-j-m-s+k)}]$$

where

$$K_{j_1 j_2 jm} = (B_{j_1 j_2 j})^{-\frac{1}{2}} (j_1+j_2-j)!\, [(2j)!]^{-\frac{1}{2}} (j_1-j_2+j)!$$
$$(j_2-j_1+j)!\, [(j+m)!(j-m)!]^{\frac{1}{2}}$$
$$= [(2j+1)(j_1+j_2-j)!(j_1-j_2+j)!(j_2-j_1+j)!]^{\frac{1}{2}}$$
$$[(j_1+j_2+j+1)!(2j_1)!(2j_2)!]^{-\frac{1}{2}}\, [(j+m)!(j-m)!]^{\frac{1}{2}}$$

Now replace the index k by $m_1 = j_2-j-s+k$, then

$$\Phi_{j_1 j_2}(j,m) = K_{j_1 j_2 jm}\, S(1\cdots 2j_1) S(2j_1+1\cdots 2j_1+2j_2) \sum_{m_1} \sum_s (-1)^s$$
$$[s!(j_1+j_2-j-s)!(j_1-m_1-s)!(j_2+m-m_1-s)!(j-j_2+m_1+s)!$$
$$(j-j_1-m+m_1+s)!]^{-1} [g_1^{\otimes(j_1+m_1)} \otimes g_2^{\otimes(j_1-m_1)}] \otimes [g_1^{\otimes(j_2+m-m_1)}$$
$$\otimes g_2^{\otimes(j_2-m+m_1)}]$$

where m_1,s are summed over all values which give rise to no factorials of negative integers in the summand. The index s has integer values. The index m_1 has integer values if j_1 is an integer and half-odd integer values if j_1 is a half-odd integer. Put $m_2 = m-m_1$, insert the Kronecker delta $\delta(m_1+m_2,m)$ and sum over both m_1 and m_2. Note that m_2 has integer values if j_2 is an integer and half-odd integer values if j_2 is a half-odd integer. We get

$$\Phi_{j_1 j_2}(j,m) = K_{j_1 j_2 jm} \sum_{m_1,m_2} \delta(m_1+m_2,m) \sum_s (-1)^s [s!(j_1+j_2-j-s)!$$

$$(j_1-m_1-s)!(j_2+m_2-s)!(j-j_2+m_1+s)!(j-j_1-m_2+s)!]^{-1}$$

$$[S(1\cdots 2j_1) g_1^{\otimes(j_1+m_1)} \otimes g_2^{\otimes(j_1-m_1)}] \otimes [S(1\cdots 2j_2)$$

$$g_1^{\otimes(j_2+m_2)} \otimes g_2^{\otimes(j_2-m_2)}]$$

$$= \sum_{m_1} \sum_{m_2} \delta(m_1+m_2,m) N_{j_1 j_2}(m_1,m_2;j,m) \sum_s (-1)^s [s!(j_1+j_2-j-s)!$$

$$(j_1-m_1-s)!(j_2+m_2-s)!(j-j_2+m_1+s)!(j-j_1-m_2+s)!]^{-1}\ \Phi(j_1,m_1) \otimes \Phi(j_2,m_2)$$

where

$$N_{j_1 j_2}(m_1,m_2;j,m) = K_{j_1 j_2 jm}[(2j_1)!(j_1+m_1)!(j_1-m_1)!$$

$$(2j_2)!(j_2+m_2)!(j_2-m_2)!]^{\frac{1}{2}}$$

$$= [(2j+1)(j_1+j_2-j)!(j_1-j_2+j)!(j_2-j_1+j)!]^{\frac{1}{2}}[(j_1+j_2+j+1)]^{-\frac{1}{2}}$$

$$[(j+m)!(j-m)!(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!]^{\frac{1}{2}}$$

From the definition of the Clebsch-Gordan coefficients, we arrive at

$$C_{j_1 j_2}(m_1,m_2;j,m) = \delta(m_1+m_2,m) N_{j_1 j_2}(m_1,m_2;j,m) \sum_s (-1)^s$$

$$[s!(j_1+j_2-j-s)!(j_1-m_1-s)!(j_2+m_2-s)!(j-j_2+m_1+s)!(j-j_1-m_2+s)!]^{-1}$$

CHAPTER 2

MAP OF EVEN SPIN-TENSORS INTO TENSORS AND FUNCTIONS ON EUCLIDEAN 3-SPACE

The Tensor Space $(U_2 \otimes \overline{U}_2)^{\otimes \ell}$

A tensor T of order 2ℓ in $(U_2 \otimes \overline{U}_2)^{\otimes \ell}$ is defined as a multilinear functional whose value

$$(T \mid \overline{w}_1, y_1, \overline{w}_2, y_2, \cdots, \overline{w}_\ell, y_\ell)$$

is linear in its arguments $\overline{w}_1, \overline{w}_2, \cdots, \overline{w}_\ell$ in $\overline{U}_2$ and $y_1, y_2, \cdots, y_\ell$ in U_2. A tensor S' of order 2ℓ in $(\overline{U}_2 \otimes U_2)^{\otimes \ell}$ is defined as a multilinear functional whose value

$$(S' \mid w_1, \overline{y}_1, w_2, \overline{y}_2, \cdots, w_\ell, \overline{y}_\ell)$$

is linear in its arguments $w_1, w_2, \cdots, w_\ell$ in U_2 and $\overline{y}_1, \overline{y}_2, \cdots, \overline{y}_\ell$ in $\overline{U}_2$. Addition of tensors and multiplication by complex numbers are defined similarly as in other tensor spaces.

For T in $(U_2 \otimes \overline{U}_2)^{\otimes \ell}$ and S' in $(\overline{U}_2 \otimes U_2)^{\otimes \ell}$, their conjugates $\overline{T}$ in $(\overline{U}_2 \otimes U_2)^{\otimes \ell}$ and $\overline{S}'$ in $(U_2 \otimes \overline{U}_2)^{\otimes \ell}$ are defined as

$$(\overline{T} \mid w_1, \overline{y}_1, \cdots, w_\ell, \overline{y}_\ell) = \overline{(T \mid \overline{w}_1, y_1, \cdots, \overline{w}_\ell, y_\ell)}$$

$$(\overline{S}' \mid \overline{w}_1, y_1, \cdots, \overline{w}_\ell, y_\ell) = \overline{(S' \mid w_1, \overline{y}_1, \cdots, w_\ell, \overline{y}_\ell)}$$

From now on we shall express elements in $(\overline{U}_2 \otimes U_2)^{\otimes \ell}$ as the conjugate of elements in $(U_2 \otimes \overline{U}_2)^{\otimes \ell}$, i.e. instead of S' we write $\overline{S}$ with S in $(U_2 \otimes \overline{U}_2)^{\otimes \ell}$.

Given $u_1, u_2, \cdots, u_\ell$ in U_2 and $\overline{v}_1, \overline{v}_2, \cdots, \overline{v}_\ell$ in $\overline{U}_2$ their tensor product

$$\bigotimes_{k=1}^{\ell} (u_k \overline{v}_k) \equiv u_1 \overline{v}_1 u_2 \overline{v}_2 \cdots u_\ell \overline{v}_\ell$$

is an element of $(U_2 \otimes \overline{U}_2)^{\otimes \ell}$ defined as

$$(\bigotimes_{k=1}^{\ell} u_k \overline{v}_k \mid \overline{w}_1, y_1, \overline{w}_2, \overline{y}_2, \cdots, \overline{w}_\ell, y_\ell) = \prod_{k=1}^{\ell} (u_k \circ \overline{w}_k)(\overline{v}_k \circ y_k)$$

for all $\overline{w}_1, \overline{w}_2, \cdots, \overline{w}_\ell$ in $\overline{U}_2$ and $y_1, y_2, \cdots, y_\ell$ in U_2, and the tensor product

$$\bigotimes_{k=1}^{\ell} (\overline{v}_k u_k) \equiv \overline{v}_1 u_1 \overline{v}_2 u_2 \cdots \overline{v}_\ell u_\ell$$

is an element of $(\overline{U}_2 \otimes U_2)^{\otimes \ell}$ defined as

$$(\bigotimes_{k=1}^{\ell} \overline{v}_k u_k \mid w_1, \overline{y}_1, w_2, \overline{y}_2, \cdots, w_\ell, \overline{y}_\ell) = \prod_{k=1}^{\ell} (\overline{v}_k \circ w_k)(u_k \circ \overline{y}_k)$$

for all $w_1, w_2, \cdots, w_\ell$ in U_2 and $\overline{y}_1, \overline{y}_2, \cdots, \overline{y}_\ell$ in $\overline{U}_2$. The conjugation operation has properties as previously listed when operating on other tensor spaces discussed and will not be repeated here. However we list the following:

If

$$T = \bigotimes_{k=1}^{\ell} (u_k \overline{v}_k)$$

then

$$\overline{T} = \bigotimes_{k=1}^{\ell} (\overline{u}_k v_k)$$

and

$$\overline{\overline{T}} = T$$

The tensor products $S \otimes T$ and $\overline{S} \otimes \overline{T}$ for S in $(U_2 \otimes \overline{U}_2)^{\otimes \ell}$ and T in $(U_2 \otimes \overline{U}_2)^{\otimes k}$ are defined similarly to previous definitions, and have similar properties.

For $\overline{S}$ in $(\overline{U}_2 \otimes U_2)^{\otimes \ell}$ and T in $(U_2 \otimes \overline{U}_2)^{\otimes \ell}$, their dot product $\overline{S} \odot T \equiv T \odot \overline{S}$ is defined in the special case where

$\overline{S} = \bigotimes_{k=1}^{\ell} (\overline{u}_k v_k)$ and $T = \bigotimes_{k=1}^{\ell} (s_k \overline{t}_k)$ as

$$(\overline{u}_1 v_1 \overline{u}_2 v_2 \cdots \overline{u}_\ell v_\ell) \odot (s_1 \overline{t}_1 s_2 \overline{t}_2 \cdots s_\ell \overline{t}_\ell)$$

$$= (\overline{u}_1 \circ s_1)(v_1 \circ \overline{t}_1)(\overline{u}_2 \circ s_2)(v_2 \circ \overline{t}_2) \cdots (\overline{u}_\ell \circ s_\ell)(v_\ell \circ \overline{t}_\ell)$$

and uniquely extended to all $\overline{S}$ and T by the assumption of linearity in $\overline{S}$ and T. The inner product $<S|T>$ for S,T in $(U_2 \otimes \overline{U}_2)^{\otimes \ell}$ is defined as

$$<S \mid T> = \overline{S} \odot T$$

The dot product and the inner product have properties as previously listed for other tensors, and in particular we repeat the property

$$(\overline{S} \otimes \overline{Q}) \odot (T \otimes R) = (\overline{S} \odot T)(\overline{Q} \odot R)$$

for S,T in $(U_2 \otimes \overline{U}_2)^{\otimes \ell}$ and Q,R in $(U_2 \otimes \overline{U}_2)^{\otimes k}$.

Permutations act on the spaces $(U_2 \otimes \overline{U}_2)^{\otimes \ell}$ and $(\overline{U}_2 \otimes U_2)^{\otimes \ell}$ in the same way as in cases previously described. For $p \varepsilon S_{2\ell}$, if pk is odd for all odd k and pk is even for all even k, then the action of p on tensors in these two spaces will give tensors in the same spaces. For $p \varepsilon S_{2\ell}$, if pk is even for all odd k and pk is odd for all even k, then the action of p on tensors in $(U_2 \otimes \overline{U}_2)^{\otimes \ell}$ will give tensors in $(\overline{U}_2 \otimes U_2)^{\otimes \ell}$ and vice versa.

The Spaces $E_3 \equiv (U_2 \otimes_H \bar{U}_2)_{tr}$ and $CE_3 \equiv (U_2 \otimes \bar{U}_2)_{tr}$

The unit tensor I_2 in $U_2 \otimes \bar{U}_2$ is defined as

$$(I_2 \mid \bar{s}_1, s_2) = \bar{s}_1 \circ s_2$$

for all s_1, s_2 in U_2. Its conjugate $\bar{I}_2$ satisfies

$$(\bar{I}_2 \mid s_1, \bar{s}_2) = s_1 \circ \bar{s}_2$$

and therefore is the unit tensor in $\bar{U}_2 \otimes U_2$.

The trace tr(X) of a tensor X in $U_2 \otimes \bar{U}_2$ is defined as

$$\mathrm{tr}(X) = X \odot \bar{I}_2$$

If tr(X)=0, the tensor X is called traceless. The transpose $\tilde{X}$ of a tensor X in $U_2 \otimes \bar{U}_2$ is a tensor in $\bar{U}_2 \otimes U_2$ defined as

$$(\tilde{X} \mid s_1, \bar{s}_2) = (X \mid \bar{s}_2, s_1)$$

for all s_1, s_2 in U_2. Similarly for $\bar{Y}$ in $\bar{U}_2 \otimes U_2$ its transpose is defined as

$$(\tilde{\bar{Y}} \mid \bar{s}_1, s_2) = (\bar{Y} \mid s_2, \bar{s}_1)$$

for all s_1, s_2 in U_2. For X in $U_2 \otimes \bar{U}_2$ its Hermitian adjoint X^+ is defined as

$$X^+ = \tilde{\bar{X}}$$

If X in $U_2 \otimes \bar{U}_2$ satisfies

$$X^+ = X$$

it is called Hermitian. If X is Hermitian, it follows that

$$\tilde{X} = \bar{X}$$

Define the dot product $X \cdot Y$ for X,Y in $U_2 \otimes \overline{U}_2$ as

$$X \cdot Y \equiv \tilde{X} \odot Y \equiv X \odot \tilde{Y}$$

The space $(U_2 \otimes_H \overline{U}_2)_{tr}$ is the set of all Hermitian, traceless tensors in $U_2 \otimes \overline{U}_2$. With the above dot product, this space is isomorphic to the 3-dimensional Euclidean space E_3. We shall regard them as identical and write $E_3 \equiv (U_2 \otimes_H \overline{U}_2)_{tr}$ and use both spin-tensor notation X and vector notation $\vec{x}$ for elements in this space.

The complexification CE_3 of E_3 is the set of complex linear combinations of elements of E_3. We shall write $CE_3 \equiv (U_2 \otimes \overline{U}_2)_{tr}$, the subset of all traceless tensors in $U_2 \otimes \overline{U}_2$, due to isomorphism of these two spaces. In CE_3 we have the dot product

$$X \cdot Y \equiv \tilde{X} \odot Y \equiv X \odot \tilde{Y}$$

and also the inner product

$$\langle X|Y\rangle = \overline{X} \odot Y = X^{+} \cdot Y$$

Note that in E_3

$$X \cdot Y = \langle X|Y\rangle$$

We shall need the following set

$$\vec{e}_1 = \frac{1}{\sqrt{2}} (g_1\overline{g}_2 + g_2\overline{g}_1)$$

$$\vec{e}_2 = \frac{i}{\sqrt{2}} (g_2\overline{g}_1 - g_1\overline{g}_2)$$

$$\vec{e}_3 = \frac{1}{\sqrt{2}} (g_1\overline{g}_1 - g_2\overline{g}_2)$$

which is a basis for E_3, and therefore is also a basis for CE_3.

The Tensor Spaces $CE_3^{\otimes\ell}$ and $E_3^{\otimes\ell}$

The space $CE_3^{\otimes\ell}$ is a subspace over C of $(U_2 \otimes \overline{U}_2)^{\otimes\ell}$ spanned by complex linear combinations of tensors of the form $\bigotimes_{i=1}^{\ell} \vec{y}_i$ with $\vec{y}_i \varepsilon\ CE_3 \equiv (U_2 \otimes \overline{U}_2)_{tr}$. The space $E_3^{\otimes\ell}$ is a subspace over R of $(U_2 \otimes \overline{U}_2)^{\otimes\ell}$ spanned by real linear combinations of tensors of the form $\bigotimes_{i=1}^{\ell} \vec{y}_i$ with $\vec{y}_i \varepsilon E_3 \equiv (U_2 \otimes_H \overline{U}_2)_{tr}$. Some operations which we shall define for $CE_3^{\otimes\ell}$ will also apply to $E_3^{\otimes\ell}$ since $E_3^{\otimes\ell}$ is a subspace over R of $CE_3^{\otimes\ell}$.

Tensors $T \varepsilon CE_3^{\otimes\ell}$ can be regarded as ℓ-linear functionals on CE_3 according to the following definition of the action of T on $\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_\ell$ in CE_3 by

$$(T \mid \vec{x}_1, \vec{x}_2, \cdots, \vec{x}_\ell) = T \odot (\tilde{\vec{x}}_1 \otimes \tilde{\vec{x}}_2 \otimes \cdots \otimes \tilde{\vec{x}}_\ell)$$

It follows that for $\vec{y}_i \varepsilon CE_3$ we have

$$(\bigotimes_{i=1}^{\ell} \vec{y}_i \mid \vec{x}_1, \vec{x}_2, \cdots, \vec{x}_\ell) = \prod_{i=1}^{\ell} (\vec{y}_i \cdot \vec{x}_i)$$

The unit tensor I_2 in $E_3 \otimes E_3$ and $(CE_3) \otimes (CE_3)$ is defined as

$$(I_2 \mid \vec{x}_1, \vec{x}_2) = \vec{x}_1 \cdot \vec{x}_2$$

for all $\vec{x}_1, \vec{x}_2$ in CE_3.

For T in $CE_3^{\otimes\ell}$, the contraction C(i,j)T of T where $i,j = 1,2,\cdots,\ell$, with $i<j$, is a tensor in $CE_3^{\otimes(\ell-2)}$ defined in a special case as

$$C(i,j)\left(\bigotimes_{k=1}^{\ell}\vec{y}_k\right) = (\vec{y}_i\cdot\vec{y}_j)\left(\bigotimes_{k=1}^{i-1}\vec{y}_k\right)\otimes\left(\bigotimes_{k=i+1}^{j-1}\vec{y}_k\right)\otimes\left(\bigotimes_{k=j+1}^{\ell}\vec{y}_k\right)$$

and extended linearly to all T in the space. A tensor T in $CE_3^{\otimes\ell}$ is said to be traceless if $C(i,j)T = 0$ for every contraction.

For S and T in $CE_3^{\otimes\ell}$, $S \boldsymbol{\circ} T$ is a scalar defined in a special case as

$$\left(\bigotimes_{i=1}^{\ell}\vec{y}_i\right)\boldsymbol{\circ}\left(\bigotimes_{i=1}^{\ell}\vec{z}_i\right) = \prod_{i=1}^{\ell}(\vec{y}_i\cdot\vec{z}_i)$$

for $\vec{y}_i,\vec{z}_i$ in CE_3 and extended linearly to all S and T in the space.

For any permutation p in S_ℓ, we define its operation on any T in $CE_3^{\otimes\ell}$ as

$$(pT\,|\,\vec{x}_1,\vec{x}_2,\cdots,\vec{x}_\ell) = (T\,|\,\vec{x}_{p1},\vec{x}_{p2},\cdots,\vec{x}_{p\ell})$$

for all $\vec{x}_1,\vec{x}_2,\cdots,\vec{x}_\ell$ in CE_3. A tensor $T\varepsilon CE_3^{\otimes\ell}$ is symmetric iff

$$pT = T$$

for all p in S_ℓ. The space $(CE_3^{\otimes\ell})_s$ is a subspace over C of $CE_3^{\otimes\ell}$ consisting of all symmetric tensors in $CE_3^{\otimes\ell}$. The space $(E_3^{\otimes\ell})_s$ is a subspace of $E_3^{\otimes\ell}$ consisting of all symmetric tensors in $E_3^{\otimes\ell}$. The projection operator onto these subspaces of symmetric tensors is $\frac{1}{\ell!}S(1\cdots\ell)$.

The space $(CE_3^{\otimes\ell})_{s,tr}$ is a subspace over C of $CE_3^{\otimes\ell}$ consisting of all the symmetric, traceless tensors in $CE_3^{\otimes\ell}$. The space $(E_3^{\otimes\ell})_{s,tr}$ is a subspace over R of $E_3^{\otimes\ell}$ consisting of all the symmetric, traceless tensors in $E_3^{\otimes\ell}$.

Linear Transformations on $U_2 \otimes \overline{U}_2$

For any linear transformation M on U_2, the linear transformations $\Gamma(M)$, $\Gamma_1(M)$, $\Gamma_2(M)$ on $U_2 \otimes \overline{U}_2$ are defined as

$$(\Gamma(M)X \mid \overline{s}_1, s_2) = (X \mid \overline{\tilde{M}s}_1, \tilde{\overline{M}}s_2)$$

$$(\Gamma_1(M)X \mid \overline{s}_1, s_2) = (X \mid \overline{\tilde{M}s}_1, s_2)$$

$$(\Gamma_2(M)X \mid \overline{s}_1, s_2) = (X \mid \overline{s}_1, \tilde{\overline{M}}s_2)$$

for all s_1, s_2 in U_2. It follows that

$$\Gamma(M)(u_1\overline{u}_2) = (Mu_1)(\overline{M}\,\overline{u}_2)$$

$$\Gamma_1(M)(u_1\overline{u}_2) = (Mu_1)\overline{u}_2$$

$$\Gamma_2(M)(u_1\overline{u}_2) = u_1(\overline{M}\,\overline{u}_2)$$

for arbitrary u_1, u_2 in U_2. Also

$$\Gamma(M) = \Gamma_1(M)\Gamma_2(M)$$

$$\Gamma(E) = \Gamma_1(E) = \Gamma_2(E)$$

We also define the linear transformation D(M) on $U_2 \otimes \overline{U}_2$ as

$$D(M) = \Gamma_1(M) + \Gamma_2(-M^+)$$

It follows that

$$D(M)(u_1\overline{u}_2) = (Mu_1)\overline{u}_2 + u_1(-\tilde{M}\,\overline{u}_2)$$

Furthermore if $M(\varepsilon)$ is a one-parameter family of SU(2) transformations on U_2 with $M(0)=E$, and

$$A = \frac{d}{d\varepsilon} M(\varepsilon)\Big|_{\varepsilon=0}$$

it follows that

$$\frac{d}{d\varepsilon}\left[\Gamma(M(\varepsilon))\right]\Big|_{\varepsilon=0} = D(A)$$

Note also the property

$$D(\alpha A+\beta B) = \alpha D(A) + \beta D(B)$$

where $\alpha,\beta \in C$.

Finally, we define

$$\Delta(AB) = D(A)D(B)$$

for A and B in su(2).

For the operators J_x, J_y, J_z which form a basis of su(2), it can be shown that

$$\tilde{J}_x = \bar{J}_x\ ,\ \tilde{J}_y = \bar{J}_y\ ,\ \tilde{J}_z = \bar{J}_z$$

i.e. they are Hermitian. It is easily shown that

$$\begin{aligned} \tilde{J}_x\bar{g}_1 &= \frac{1}{2}\bar{g}_2 \ , & \tilde{J}_x\bar{g}_2 &= \frac{1}{2}\bar{g}_1 \\ \tilde{J}_y\bar{g}_1 &= -\frac{i}{2}\bar{g}_2 \ , & \tilde{J}_y\bar{g}_2 &= \frac{i}{2}\bar{g}_1 \\ \tilde{J}_z\bar{g}_1 &= \frac{1}{2}\bar{g}_1 \ , & \tilde{J}_z\bar{g}_2 &= -\frac{1}{2}\bar{g}_2 \end{aligned}$$

For the basis $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ of $E_3 \equiv (U_2 \otimes_H \bar{U}_2)_{tr}$, it is easy to derive the following

$$D(-iJ_x)\vec{e}_1 = D(-iJ_y)\vec{e}_2 = D(-iJ_z)\vec{e}_3 = \vec{0}$$

$$D(-iJ_x)\vec{e}_2 = \vec{e}_3$$

$$D(-iJ_x)\vec{e}_3 = -\vec{e}_2$$

$$D(-iJ_y)\vec{e}_1 = -\vec{e}_3$$

$$D(-iJ_y)\vec{e}_3 = \vec{e}_1$$

$$D(-iJ_z)\vec{e}_1 = \vec{e}_2$$

$$D(-iJ_z)\vec{e}_2 = -\vec{e}_1$$

and

$$\Delta(J^2)\vec{e}_1 = 2\vec{e}_1$$

$$\Delta(J^2)\vec{e}_2 = 2\vec{e}_2$$

$$\Delta(J^2)\vec{e}_3 = 2\vec{e}_3 .$$

Comparing this with the vector product of two vectors in E_3, we can write

$$D(-iJ_x)\vec{r} = \vec{e}_1 \times \vec{r}$$

$$D(-iJ_y)\vec{r} = \vec{e}_2 \times \vec{r}$$

$$D(-iJ_z)\vec{r} = \vec{e}_3 \times \vec{r}$$

for arbitrary $\vec{r}$ in E_3.

If we define

$$\vec{e}_+ = \frac{1}{\sqrt{2}}(\vec{e}_1 + i\vec{e}_2)$$

$$\vec{e}_- = \frac{1}{\sqrt{2}}(\vec{e}_1 - i\vec{e}_2)$$

we see that $\vec{e}_+$, $\vec{e}_-$, $\vec{e}_3$ are a set of simultaneous eigenvectors of $D(J_z)$ and $\Delta(J^2)$ satisfying

$$D(J_z)\vec{e}_+ = \vec{e}_+$$

$$D(J_z)\vec{e}_3 = \vec{0}$$

$$D(J_z)\vec{e}_- = -\vec{e}_-$$

$$\Delta(J^2)\vec{e}_+ = 2\vec{e}_+$$

$$\Delta(J^2)\vec{e}_3 = 2\vec{e}_3$$

$$\Delta(J^2)\vec{e}_- = 2\vec{e}_-$$

We can also derive

$$D(J_+)\vec{e}_+ = \vec{0}$$

$$D(J_+)\vec{e}_3 = -\sqrt{2}\,\vec{e}_+$$

$$D(J_+)\vec{e}_- = \sqrt{2}\,\vec{e}_3$$

$$D(J_-)\vec{e}_+ = -\sqrt{2}\,\vec{e}_3$$

$$D(J_-)\vec{e}_3 = \sqrt{2}\,\vec{e}_-$$

$$D(J_-)\vec{e}_- = \vec{0}$$

thus $D(J_+)$ and $D(J_-)$ step up and step down the eigenvalues when operating on eigentensors.

The Map $I_2 \otimes \overline{\Lambda}$ from $U_2 \otimes U_2$ to $U_2 \otimes \overline{U}_2$

Corresponding to the unit tensor I_2 in $U_2 \otimes \overline{U}_2$ is the unit map I_2; $U_2 \to U_2$ (denoted by the same symbol as the tensor) defined to satisfy the equation

$$\overline{u} \circ (I_2 v) = (I_2 | \, \overline{u}, v)$$

for all u,v in U_2. Corresponding to the unit tensor $\overline{I}_2$ is the unit map $\overline{I}_2$; $\overline{U}_2 \to \overline{U}_2$ defined to satisfy the equation

$$u \circ (\overline{I}_2 \overline{v}) = (\overline{I}_2 | \, u, \overline{v})$$

for all u,v in U_2. It follows that

$$I_2 u = u$$

$$\overline{I}_2 \overline{v} = \overline{v}$$

for all u,v in U_2.

Corresponding to the antisymmetric tensor Λ in $U_2 \otimes U_2$ is the map $\Lambda; \bar{U}_2 \rightarrow U_2$ defined to satisfy the equation

$$\bar{u} \circ (\Lambda\bar{v}) = (\Lambda|\bar{u},\bar{v})$$

for all u,v in U_2. Corresponding to the tensor $\bar{\Lambda}$ in $\bar{U}_2 \otimes \bar{U}_2$ is the map $\bar{\Lambda}; U_2 \rightarrow \bar{U}_2$ defined to satisfy the equation

$$u \circ (\bar{\Lambda}v) = (\bar{\Lambda} \mid u,v)$$

for all u,v in U_2. It follows that

$$\Lambda\bar{g}_1 = -g_2, \quad \Lambda\bar{g}_2 = g_1$$

$$\bar{\Lambda}g_1 = -\bar{g}_2, \quad \bar{\Lambda}g_2 = \bar{g}_1$$

and that

$$\bar{\Lambda}u = \overline{\Lambda\bar{u}}$$

$$\tilde{\Lambda} = - \Lambda$$

$$\tilde{\bar{\Lambda}} = - \bar{\Lambda}$$

for these maps. The following properties can be proved:

$$\Lambda\bar{\Lambda} = -I_2$$

$$\bar{\Lambda}\Lambda = -\bar{I}_2$$

The tensor product map $I_2 \otimes \bar{\Lambda}; U_2 \otimes U_2 \rightarrow U_2 \otimes \bar{U}_2$ is defined on uv by

$$(I_2 \otimes \bar{\Lambda})(uv) = (I_2u)(\bar{\Lambda}\ v) = u(\bar{\Lambda}\ v)$$

and extended linearly to operate on all tensors in $U_2 \otimes U_2$. The map $I_2 \otimes \Lambda; U_2 \otimes \bar{U}_2 \rightarrow U_2 \otimes U_2$ is defined on $u\bar{v}$ by

$$(I_2 \otimes \Lambda)(u\bar{v}) = (I_2u)(\Lambda\bar{v}) = u(\Lambda\bar{v})$$

and extended linearly to operate on all tensors in $U_2 \otimes \overline{U}_2$. It follows that

$$(I_2 \otimes \overline{\Lambda})(-I_2 \otimes \Lambda) = I_2 \otimes \overline{I}_2$$

$$(-I_2 \otimes \Lambda)(I_2 \otimes \overline{\Lambda}) = I_2 \otimes I_2$$

where $I_2 \otimes \overline{I}_2$ is the unit map on $U_2 \otimes \overline{U}_2$ and $I_2 \otimes I_2$ is the unit map on $U_2 \otimes U_2$. Similarly the maps $\overline{\Lambda} \otimes I_2$; $U_2 \otimes U_2 \to \overline{U}_2 \otimes U_2$, $\Lambda \otimes I_2$; $\overline{U}_2 \otimes U_2 \to U_2 \otimes U_2$, and $(\overline{\Lambda} \otimes \Lambda)$; $U_2 \otimes \overline{U}_2 \to \overline{U}_2 \otimes U_2$ can be defined.

The following theorem can be proved.

Theorem 1: For S, T in $U_2 \otimes U_2$

$$\langle S|T\rangle = \langle (I_2 \otimes \overline{\Lambda})S \mid (I_2 \otimes \overline{\Lambda})T\rangle$$

i.e., the map $I_2 \otimes \overline{\Lambda}$; $U_2 \otimes U_2 \to U_2 \otimes \overline{U}_2$ preserves inner products. Likewise the map $I_2 \otimes \Lambda$; $U_2 \otimes \overline{U}_2 \to U_2 \otimes U_2$ preserves inner products.

Theorem 2: When operating on $U_2 \otimes U_2$ we have

$$D(J_x)(I_2 \otimes \overline{\Lambda}) = (I_2 \otimes \overline{\Lambda})D(J_x)$$

$$D(J_y)(I_2 \otimes \overline{\Lambda}) = (I_2 \otimes \overline{\Lambda})D(J_y)$$

$$D(J_z)(I_2 \otimes \overline{\Lambda}) = (I_2 \otimes \overline{\Lambda})D(J_z)$$

hence

$$D(A)\ (I_2 \otimes \overline{\Lambda}) = (I_2 \otimes \overline{\Lambda})D(A)$$

for arbitrary A in su(2).

Proof: We have shown that

$$D(A)\Lambda = 0$$

for A in su(2). Therefore for arbitrary u,v in U_2 we have

$$0 = (D(A)\Lambda \mid \bar{u},\bar{v})$$

$$= (\Lambda \mid \tilde{A}\ \bar{u},\bar{v}) + (\Lambda \mid \bar{u},\tilde{A}\ \bar{v})$$

$$= (\tilde{A}\ \bar{u}) \circ (\Lambda\bar{v}) + \bar{u} \circ (\Lambda\tilde{A}\ \bar{v})$$

$$= \bar{u} \circ (A\Lambda\bar{v}) + \bar{u} \circ (\Lambda\tilde{A}\ \bar{v})$$

therefore

$$A\Lambda = -\Lambda\tilde{A}$$

Then

$$D(A)(I_2 \otimes \Lambda)(u\bar{v})$$

$$= D(A)[u(\Lambda\bar{v})]$$

$$= (Au)(\Lambda\bar{v}) + u(A\Lambda\bar{v})$$

$$= (Au)(\Lambda\bar{v}) + u(-\Lambda\tilde{A}\ \bar{v})$$

$$= (I_2 \otimes \Lambda)[(Au)\bar{v} - u(\tilde{A}\ \bar{v})]$$

$$= (I_2 \otimes \Lambda)D(A)(u\bar{v})$$

Therefore

$$D(A)(I_2 \otimes \Lambda) = (I_2 \otimes \Lambda)D(A)$$

Then multiplying both sides on the left and right by $-I_2 \otimes \bar{\Lambda}$ gives

$$(I_2 \otimes \bar{\Lambda})D(A) = D(A)(I_2 \otimes \bar{\Lambda}) \qquad \text{Q.E.D.}$$

Theorem 3: Under the map $I_2 \otimes \bar{\Lambda}$; $U_2 \otimes U_2 \rightarrow U_2 \otimes \bar{U}_2$, the subspace $(U_2 \otimes U_2)_s$ is mapped one-to-one onto the subspace $(U_2 \otimes \bar{U}_2)_{tr} \equiv CE_3$.

For the simultaneous eigenstates of $D(J_z)$ and $\Delta(J^2)$ in $U_2 \otimes U_2$

$$\Phi(1,1) = g_1g_1$$

$$\Phi(1,-1) = g_2g_2$$

$$\Phi(1,0) = \frac{1}{\sqrt{2}}(g_1g_2+g_2g_1)$$

the map $I_2 \otimes \overline{\Lambda}$ gives the simultaneous eigenstates of $D(J_z)$ and $\Delta(J^2)$ in $U_2 \otimes \overline{U}_2$:

$$(I_2 \otimes \overline{\Lambda})\ \Phi(1,1) = -\vec{e}_+$$

$$(I_2 \otimes \overline{\Lambda})\ \Phi(1,-1) = \vec{e}_-$$

$$(I_2 \otimes \overline{\Lambda})\ \Phi(1,0) = \vec{e}_3$$

It can also be shown that for $\vec{w}$ in E_3 or CE_3

$$\tilde{\vec{w}} = -(\overline{\Lambda} \otimes \Lambda)\vec{w}$$

Consequently for $\vec{v}$ and $\vec{w}$ in E_3 or CE_3 we have

$$\vec{v} \cdot \vec{w} = -\vec{v} \odot [(\overline{\Lambda} \otimes \Lambda)\vec{w}] = -[(\overline{\Lambda} \otimes \Lambda)\vec{v}] \odot \vec{w}$$

$$= [(\overline{\Lambda} \otimes \overline{I}_2)\vec{v}] \odot [I_2 \otimes \Lambda)\vec{w}]$$

$$= [(I_2 \otimes \Lambda)\vec{v}] \odot [(\overline{\Lambda} \otimes \overline{I}_2)\vec{w}]$$

The Map $(I_2 \otimes \overline{\Lambda})^{\otimes \ell}$ and Linear Transformations on $(U_2 \otimes \overline{U}_2)^{\otimes \ell}$

We can generalize to the one-to-one onto map $(I_2 \otimes \overline{\Lambda})^{\otimes \ell}$; $U_2^{\otimes 2\ell} \to (U_2 \otimes \overline{U}_2)^{\otimes \ell}$ with $(I_2 \otimes \overline{\Lambda})^{\otimes \ell}$ being defined straightforwardly by

$$(I_2 \otimes \overline{\Lambda})^{\otimes \ell}\left(\bigotimes_{i=1}^{\ell} u_i v_i\right) = \bigotimes_{i=1}^{\ell} (I_2 u_i)(\overline{\Lambda}\, v_i)$$

for arbitrary tensors of the form $\bigotimes_{i=1}^{\ell} u_i v_i$ in $U_2^{\otimes 2\ell}$, and extended linearly to all of $U_2^{\otimes 2\ell}$.

Also, for T in $(U_2 \otimes \overline{U}_2)^{\otimes \ell}$, define the linear transformations $\Gamma(M)$, $\Gamma_o(M)$, $\Gamma_E(M)$, $\Gamma_{2k-1}(M)$, $\Gamma_{2k}(M)$ respectively as follows

$$(\Gamma(M)T \mid \bar{s}_1, t_1, \bar{s}_2, t_2, \cdots, \bar{s}_\ell, t_\ell)$$

$$= (T \mid \tilde{M}\,\bar{s}_1, \tilde{\bar{M}}\,t_1, \tilde{M}\,\bar{s}_2, \tilde{\bar{M}}\,t_2, \cdots, \tilde{M}\,\bar{s}_\ell, \tilde{\bar{M}}\,t_\ell)$$

$$(\Gamma_o(M)T \mid \bar{s}_1, t_1, \bar{s}_2, t_2, \cdots, \bar{s}_\ell, t_\ell)$$

$$= (T \mid \tilde{M}\,\bar{s}_1, t_1, \tilde{M}\,\bar{s}_2, t_2, \cdots, \tilde{M}\,\bar{s}_\ell, t_\ell)$$

$$(\Gamma_E(M)T \mid \bar{s}_1, t_1, \bar{s}_2, t_2, \cdots, \bar{s}_\ell, t_\ell)$$

$$= (T \mid \bar{s}_1, \tilde{\bar{M}}\,t_1, \bar{s}_2, \tilde{\bar{M}}\,t_2, \cdots, \bar{s}_\ell, \tilde{\bar{M}}\,t_\ell)$$

$$(\Gamma_{2k-1}(M)T \mid \bar{s}_1, t_1, \bar{s}_2, t_2, \cdots, \bar{s}_{k-1}, t_{k-1}, \bar{s}_k, \bar{t}_k, \bar{s}_{k+1}, t_{k+1}, \cdots, \bar{s}_\ell, t_\ell)$$

$$= (T \mid \bar{s}_1, t_1, \bar{s}_2, t_2, \cdots, \bar{s}_{k-1}, t_{k-1}, \tilde{M}\,\bar{s}_k, t_k, \bar{s}_{k+1}, t_{k+1}, \cdots, \bar{s}_\ell, t_\ell)$$

$$(\Gamma_{2k}(M)T \mid \bar{s}_1, t_1, \bar{s}_2, t_2, \cdots, \bar{s}_{k-1}, t_{k-1}, \bar{s}_k, t_k, \bar{s}_{k+1}, t_{k+1}, \cdots, \bar{s}_\ell, t_\ell)$$

$$= (T \mid \bar{s}_1, t_1, \bar{s}_2, t_2, \cdots, \bar{s}_{k-1}, t_{k-1}, \bar{s}_k, \tilde{\bar{M}}\,t_k, \bar{s}_{k+1}, t_{k+1}, \cdots, \bar{s}_\ell)t_\ell)$$

for arbitrary s_i, t_i ; $i = 1,2,\cdots,\ell$ in U_2. It follows that

$$\Gamma(M) = \Gamma_o(M)\Gamma_E(M)$$

$$\Gamma(E) = \Gamma_o(E) = \Gamma_E(E) = \Gamma_{2k-1}(E) = \Gamma_{2k}(E)$$

$$\Gamma_o(M) = \prod_{k=1}^{\ell} \Gamma_{2k-1}(M)$$

$$\Gamma_E(M) = \prod_{k=1}^{\ell} \Gamma_{2k}(M)$$

and

$$\Gamma(M)\Big(\bigotimes_{i=1}^{\ell} u_i\bar{v}_i\Big) = \bigotimes_{i=1}^{\ell} [(Mu_i)(\overline{Mv_i})]$$

$$\Gamma_o(M)\Big(\bigotimes_{k=1}^{\ell} u_i\bar{v}_i\Big) = \bigotimes_{i=1}^{\ell} [(Mu_i)\bar{v}_i]$$

$$\Gamma_E(M)\Big(\bigotimes_{i=1}^{\ell} u_i\bar{v}_i\Big) = \bigotimes_{i=1}^{\ell} [u_i(\overline{Mv_i})]$$

$$\Gamma_{2k-1}(M)\ \Big(\bigotimes_{k=1}^{\ell} u_i\bar{v}_i\Big) = \Big[\bigotimes_{i=1}^{k-1} (u_i\bar{v}_i)\Big] \otimes (Mu_k) \otimes \bar{v}_k \otimes \Big[\bigotimes_{i=k+1}^{\ell} (u_i\bar{v}_i)\Big]$$

$$\Gamma_{2k}(M)\Big(\bigotimes_{i=1}^{\ell} u_i\bar{v}_i\Big) = \Big[\bigotimes_{i=1}^{k-1} (u_i\bar{v}_i)\Big] \otimes u_k \otimes (\overline{Mv}_k) \otimes \Big[\bigotimes_{i=k+1}^{\ell} (u_i\bar{v}_i)\Big]$$

Define the linear transformation $D(M)$ on T in $(U_2 \otimes \bar{U}_2)^{\otimes \ell}$ by

$$D(M)T = \sum_{k=1}^{\ell} \Gamma_{2k-1}(M)T + \sum_{k=1}^{\ell} \Gamma_{2k}(-M^{+})T$$

thus

$$D(M)\Big(\bigotimes_{i=1}^{\ell} u_i\bar{v}_i\Big) = \sum_{k=1}^{\ell} \Big\{\Big[\bigotimes_{i=1}^{k-1} (u_i\bar{v}_i)\Big] \otimes (Mu_k) \otimes \bar{v}_k \otimes$$

$$\Big[\bigotimes_{i=k+1}^{\ell} (u_i\bar{v}_i)\Big]\Big\} + \sum_{k=1}^{\ell} \Big\{\Big[\bigotimes_{i=1}^{k-1} (u_i\bar{v}_i)\Big] \otimes u_k \otimes (-\tilde{M}\bar{v}_k) \otimes \Big[\bigotimes_{i=k+1}^{\ell} (u_i\bar{v}_i)\Big]\Big\}$$

Finally, we can define

$$\Delta(AB)T = D(A)[D(B)T]$$

for T in $(U_2 \otimes \bar{U}_2)^{\otimes \ell}$ and A,B in su(2).

Theorem 1:

$$D(M)(I_2 \otimes \bar{\Lambda})^{\otimes \ell} = (I_2 \otimes \bar{\Lambda})^{\otimes \ell} D(M)$$

for $M = J_x, J_y$ and J_z .

Proof:

$$D(M)\Big\{(I_2 \otimes \bar{\Lambda})^{\otimes \ell}\Big[\bigotimes_{i=1}^{\ell} (u_i v_i)\Big]\Big\}$$

$$= D(M)\Big\{\bigotimes_{i=1}^{\ell} [(I_2 \otimes \bar{\Lambda})(u_i v_i)]\Big\}$$

$$= \bigotimes_{i=1}^{\ell} \{D(M)[(I_2 \otimes \bar{\Lambda})(u_i v_i)]\}$$

$$= \bigotimes_{i=1}^{\ell} \{(I_2 \otimes \bar{\Lambda})[D(M)(u_i\bar{v}_i)]\}$$

$$= (I_2 \otimes \overline{\Lambda})^{\otimes \ell}\{\bigotimes_{i=1}^{\ell} [D(M)(u_i\overline{v}_i)]$$

$$= (I_2 \otimes \overline{\Lambda})^{\otimes \ell}\{D(M)[\bigotimes_{i=1}^{\ell} (u_i\overline{v}_i)]\}$$

for arbitrary $\bigotimes_{i=1}^{\ell} (u_i\overline{v}_i)$ in $(U_2 \otimes \overline{U}_2)^{\otimes \ell}$. This result can be extended linearly to all tensors in $(U_2 \otimes \overline{U}_2)^{\otimes \ell}$. Q.E.D.

Corollary 1:

$$\Delta(J^2)(I_2 \otimes \overline{\Lambda})^{\otimes \ell} = (I_2 \otimes \overline{\Lambda})^{\otimes \ell}\Delta(J^2)$$

Now, for the even values $\ell=0,1,2,\cdots$; $m=-\ell,-\ell+1,\cdots,0,\cdots,\ell$. We find that $(I_2 \otimes \overline{\Lambda})^{\otimes \ell}\Phi(\ell,m)$ are the eigenfunctions of $D(J_z)$ with eigenvalues m, and the eigenfunction of $\Delta(J^2)$ with eigenvalues $j(j+1)$ since by the above Theorem 1 and Corollary 1:

$$D(J_z)[(I_2 \otimes \overline{\Lambda})^{\otimes \ell}\Phi(\ell,m)] = (I_2 \otimes \overline{\Lambda})^{\otimes \ell}[D(J_z)\Phi(\ell,m)]$$

$$= m[(I_2 \otimes \overline{\Lambda})^{\otimes \ell}\Phi(\ell,m)]$$

$$\Delta(J^2)[(I_2 \otimes \overline{\Lambda})^{\otimes \ell}\Phi(\ell,m)] = (I_2 \otimes \overline{\Lambda})^{\otimes \ell}[\Delta(J^2)\Phi(\ell,m)]$$

$$= j(j+1)[(I_2 \otimes \overline{\Lambda})^{\otimes \ell}\Phi(\ell,m)]$$

Theorem 2: The map $(I_2 \otimes \overline{\Lambda})^{\otimes \ell}$; $U_2^{\otimes 2\ell} \rightarrow (U_2 \otimes \overline{U}_2)^{\otimes \ell}$ preserves inner products, i.e., for S,T in $U_2^{\otimes 2\ell}$,

$$\langle S|T\rangle = \langle (I_2 \otimes \overline{\Lambda})^{\otimes \ell} S \mid (I_2 \otimes \overline{\Lambda})^{\otimes \ell} T\rangle$$

Theorem 3: Under the map $(I_2 \otimes \overline{\Lambda})^{\otimes \ell}$; $U_2^{\otimes 2\ell} \rightarrow (U_2 \otimes \overline{U}_2)^{\otimes \ell}$ the subspace $L_{2\ell} \equiv (U_2^{\otimes 2\ell})_s$ is mapped one-to-one onto the subspace $(CE_3^{\otimes \ell})_{s,tr}$.

Corollary 1: $(I_2 \otimes \overline{\Lambda})^{\otimes \ell}\Phi(\ell,m)$ where $\ell=0,1,2,\cdots$; and $m=-\ell,-\ell+1,\cdots,0,\cdots,\ell$, is a basis for $(CE_3^{\otimes \ell})_{s,tr}$.

Corollary 2: The subspace $(CE_3^{\otimes \ell})_{s,tr}$ has dimension $2\ell+1$.

An Orthogonal Basis of $(CE_3^{\otimes \ell})_{s,tr}$

We shall give another procedure for constructing a basis for $(CE_3^{\otimes \ell})_{s,tr}$ which are simultaneous eigentensors of $D(J_z)$ and $\Delta(J^2)$. For $m=-\ell,-\ell+1,\cdots,0,\cdots,\ell$, define

$$T_{m,E} = \frac{1}{\ell!} S(1\cdots\ell)[\vec{e}_+^{\,\otimes \frac{1}{2}(\ell+m)} \otimes \vec{e}_-^{\,\otimes \frac{1}{2}(\ell-m)}]$$

if $\ell+m$ is even. And

$$T_{m,o} = \frac{1}{\ell!} S(1\cdots\ell)[\vec{e}_+^{\,\otimes \frac{1}{2}(\ell+m)} \otimes \vec{e}_-^{\,\otimes \frac{1}{2}(\ell-m)} \otimes \vec{e}_3]$$

if $\ell+m$ is odd.

Theorem 1: If $i+j+k=\ell$; i,j,k nonnegative integers then

$$D(J_z)(\vec{e}_+^{\,\otimes i} \otimes \vec{e}_-^{\,\otimes j} \otimes \vec{e}_3^{\,\otimes k}) = (i-j)(\vec{e}_+^{\,\otimes i} \otimes \vec{e}_-^{\,\otimes j} \otimes \vec{e}_3^{\,\otimes k})$$

In particular

$$D(J_z)(T_{m,E}) = mT_{m,E}$$

$$D(J_z)(T_{m,o}) = mT_{m,o}$$

The procedure is to add to $T_{m,E}$ and $T_{m,o}$ a linear combination of eigentensors of $D(J_z)$ with eigenvalue m in $(CE_3^{\otimes \ell})_s$ and then find the coefficients that will make it traceless.

Write

$$\begin{aligned}
T^{\ell}_{m,E} = \frac{1}{\ell!}\, S(1\cdots\ell)\Bigg\{ & [\vec{e}_{+}^{\,\otimes\frac{1}{2}(\ell+m)} \otimes \vec{e}_{-}^{\,\otimes\frac{1}{2}(\ell-m)}] - \alpha_1 [\vec{e}_{+}^{\,\otimes\frac{1}{2}(\ell+m-2)} \otimes \vec{e}_{-}^{\,\otimes\frac{1}{2}(\ell-m-2)} \\
& \otimes \vec{e}_{3}^{\,\otimes 2}] - \alpha_2 [\vec{e}_{+}^{\,\otimes\frac{1}{2}(\ell+m-4)} \otimes \vec{e}_{-}^{\,\otimes\frac{1}{2}(\ell-m-4)} \otimes \vec{e}_{3}^{\,\otimes 4}] - \cdots - \alpha_i [\vec{e}^{\,\otimes\frac{1}{2}(\ell+m-2i)} \\
& \otimes \vec{e}_{-}^{\,\otimes\frac{1}{2}(\ell-m-2i)} \otimes \vec{e}_{3}^{\,\otimes 2i}] - \cdots \\
& - \begin{cases} \alpha_{\frac{1}{2}(\ell-m)} \, [\vec{e}_{+}^{\,\otimes m} \otimes \vec{e}_{3}^{\,\otimes(\ell-m)}] & \text{if } m \geq 0 \\ \alpha_{\frac{1}{2}(\ell+m)} \, [\vec{e}_{-}^{\,\otimes(-m)} \otimes \vec{e}_{3}^{\,\otimes(\ell+m)}] & \text{if } m < 0 \end{cases} \Bigg\}
\end{aligned}$$

$$\begin{aligned}
T^{\ell}_{m,o} = \frac{1}{\ell!}\, S(1\cdots\ell)\Bigg\{ & [\vec{e}_{+}^{\,\otimes\frac{1}{2}(\ell+m-1)} \otimes \vec{e}_{-}^{\,\otimes\frac{1}{2}(\ell-m-1)} \otimes \vec{e}_{3}] \\
& - \beta_1 [\vec{e}_{+}^{\,\otimes\frac{1}{2}(\ell+m-3)} \otimes \vec{e}_{-}^{\,\otimes\frac{1}{2}(\ell-m-3)} \otimes \vec{e}_{3}^{\,\otimes 3}] - \beta_2 [\vec{e}_{+}^{\,\otimes\frac{1}{2}(\ell+m-5)} \otimes \vec{e}_{-}^{\,\otimes\frac{1}{2}(\ell-m-5)} \\
& \otimes \vec{e}_{3}^{\,\otimes 5}] - \cdots - \beta_i [\vec{e}_{+}^{\,\otimes\frac{1}{2}(\ell+m-1-2i)} \otimes \vec{e}_{-}^{\,\otimes\frac{1}{2}(\ell-m-1-2i)} \otimes \vec{e}_{3}^{\,\otimes(1+2i)}] - \cdots \\
& - \begin{cases} \beta_{\frac{1}{2}(\ell-m-1)} \, [\vec{e}_{+}^{\,\otimes m} \otimes \vec{e}_{3}^{\,\otimes(\ell-m)}] & \text{if } m \geq 0 \\ \beta_{\frac{1}{2}(\ell+m-1)} \, [\vec{e}_{-}^{\,\otimes(-m)} \otimes \vec{e}_{3}^{\,\otimes(\ell+m)}] & \text{if } m < 0 \end{cases} \Bigg\}
\end{aligned}$$

Theorem 2: If $C(1,2)\, T^{\ell}_{m,E} = 0$ and $C(1,2)\, T^{\ell}_{m,o} = 0$, then

$$\alpha_i = (-1)^{i-1} \frac{[(\frac{\ell+m}{2})!(\frac{\ell-m}{2})!]2^i}{(2i)![(\frac{\ell+m-2i}{2})!(\frac{\ell-m-2i}{2})!]}$$

for $i=1,2,\cdots,\min\{\frac{1}{2}(\ell-m), \frac{1}{2}(\ell+m)\}$. And,

$$\beta_i = (-1)^{i-1} \frac{[(\frac{\ell+m-1}{2})!(\frac{\ell-m-1}{2})!]2^i}{(2i+1)![(\frac{\ell+m-1-2i}{2})!(\frac{\ell-m-1-2i}{2})!]}$$

for $i=1,2,\cdots, \min\{\frac{\ell-m-1}{2}, \frac{\ell+m-1}{2}\}$.

Before proving the main theorem, let us prove the following lemmas:

Lemma 1:

$$C(1,2)S(1\cdots\ell)[\vec{e}_+^{\,\otimes\frac{1}{2}(\ell+m)} \otimes \vec{e}_-^{\,\otimes\frac{1}{2}(\ell-m)}]$$

$$= \frac{1}{2}(\ell+m)(\ell-m)S(1\cdots\ell-2)[\vec{e}_+^{\,\otimes\frac{1}{2}(\ell+m-2)} \otimes \vec{e}_-^{\,\otimes\frac{1}{2}(\ell-m-2)}]$$

Proof: By using the identity

$$S(1\cdots\ell)[\vec{a}_1 \otimes \vec{a}_2 \otimes \cdots \otimes \vec{a}_\ell]$$

$$= \vec{a}_1 \otimes [S(1\cdots\ell-1)(\vec{a}_2 \otimes \vec{a}_3 \otimes \cdots \otimes \vec{a}_\ell)] + \vec{a}_2 \otimes [S(1\cdots\ell-1)$$

$$(\vec{a}_1 \otimes \vec{a}_3 \otimes \cdots \otimes \vec{a}_\ell)] + \cdots + \vec{a}_\ell \otimes [S(1\cdots\ell-1)(\vec{a}_1 \otimes \vec{a}_2 \otimes \cdots \otimes \vec{a}_{\ell-1})]$$

we obtain

$$S(1\cdots\ell)[\vec{e}_+^{\,\otimes\frac{1}{2}(\ell+m)} \otimes \vec{e}_-^{\,\otimes\frac{1}{2}(\ell-m)}]$$

$$= \frac{1}{2}(\ell+m)\ \vec{e}_+ \otimes [S(1\cdots\ell-1)(\vec{e}_+^{\,\otimes\frac{1}{2}(\ell+m-2)} \otimes \vec{e}_-^{\,\otimes\frac{1}{2}(\ell-m)})] + \frac{1}{2}(\ell-m)\ \vec{e}_- \otimes$$

$$[S(1\cdots\ell-1)(\vec{e}_+^{\,\otimes\frac{1}{2}(\ell+m)} \otimes \vec{e}_-^{\,\otimes\frac{1}{2}(\ell-m-2)})]$$

$$= \frac{1}{2}\ (\ell+m)\vec{e}_+ \otimes \{\frac{1}{2}(\ell+m-2)\vec{e}_+ \otimes [S(1\cdots\ell-2)(\vec{e}_+^{\,\otimes\frac{1}{2}(\ell+m-4)} \otimes \vec{e}_-^{\,\otimes\frac{1}{2}(\ell-m)})]$$

$$+ \frac{1}{2}(\ell-m)\vec{e}_- \otimes [S(1\cdots\ell-2)(\vec{e}_+^{\,\otimes\frac{1}{2}(\ell+m-2)} \otimes \vec{e}_-^{\,\otimes\frac{1}{2}(\ell-m-2)})]\}$$

$$+ \frac{1}{2}(\ell-m)\vec{e}_- \otimes \{\tfrac{1}{2}(\ell+m)\vec{e}_+ \otimes [S(1\cdots\ell-2)(\vec{e}_+^{\,\otimes\frac{1}{2}(\ell+m-2)} \otimes \vec{e}_-^{\,\otimes\frac{1}{2}(\ell-m-2)})]$$

$$+ \frac{1}{2}(\ell-m)\vec{e}_- \otimes [S(1\cdots\ell-2)(\vec{e}_+^{\,\otimes\frac{1}{2}(\ell+m)} \otimes \vec{e}_-^{\,\otimes\frac{1}{2}(\ell-m-4)})]\}$$

thus

$$C(1,2)S(1\cdots\ell)[\vec{e}_+^{\,\otimes\frac{1}{2}(\ell+m)} \otimes \vec{e}_-^{\,\otimes\frac{1}{2}(\ell-m)}]$$

$$= \frac{1}{2}(\ell+m)(\ell-m)S(1\cdots\ell-2)(\vec{e}_+^{\,\otimes\frac{1}{2}(\ell+m-2)} \otimes \vec{e}_-^{\,\otimes\frac{1}{2}(\ell-m-2)}) \qquad \text{Q.E.D.}$$

Lemma 2:

$$C(1,2)S(1\cdots\ell)[\vec{e}_+^{\,\otimes\frac{1}{2}(\ell+m-2i)} \otimes \vec{e}_-^{\,\otimes\frac{1}{2}(\ell-m-2i)} \otimes \vec{e}_3^{\,\otimes 2i}]$$

$$= \frac{1}{2}(\ell+m-2i)S(1\cdots\ell-2)[\vec{e}_+^{\,\otimes\frac{1}{2}(\ell+m-2i-2)} \otimes \vec{e}_-^{\,\otimes\frac{1}{2}(\ell-m-2i-2)} \otimes \vec{e}_3^{\,\otimes 2i}]$$

$$+2i(2i-1)S(1\cdots\ell-2)[\vec{e}_+^{\,\otimes\frac{1}{2}(\ell+m-2i)} \otimes \vec{e}_-^{\,\otimes\frac{1}{2}(\ell-m-2i)} \otimes \vec{e}_3^{\,\otimes(2i-2)}]$$

for $i=1,2,\cdots, \min\{\frac{1}{2}(\ell-m), \frac{1}{2}(\ell+m)\}$.

Proof: We omit the proof. Q.E.D.

Proof of the main theorem: We require traceless tensors, hence

$$C(1,2)T^{\ell}_{m,E} = \frac{1}{\ell!}\{\tfrac{1}{2}(\ell+m)(\ell-m)S(1\cdots\ell-2)[\vec{e}_+^{\,\otimes\frac{1}{2}(\ell+m-2)} \otimes \vec{e}_-^{\,\otimes\frac{1}{2}(\ell-m-2)}]$$

$$-\alpha_1[\tfrac{1}{2}(\ell+m-2)(\ell-m-2)S(1\cdots\ell-2)(\vec{e}_+^{\,\otimes\frac{1}{2}(\ell+m-4)} \otimes \vec{e}_-^{\,\otimes\frac{1}{2}(\ell-m-4)} \otimes \vec{e}_3^{\,\otimes 2})$$

$$+ 2S(1\cdots\ell-2)(\vec{e}_+^{\,\otimes\frac{1}{2}(\ell+m-2)} \otimes \vec{e}_-^{\,\otimes\frac{1}{2}(\ell-m-2)})]-\alpha_2[\tfrac{1}{2}(\ell+m-4)(\ell-m-4)S(1\cdots\ell-2)$$

$$(\vec{e}_+^{\,\otimes\frac{1}{2}(\ell+m-6)} \otimes \vec{e}_-^{\,\otimes\frac{1}{2}(\ell-m-6)} \otimes \vec{e}_3^{\,\otimes 4}) + 4\cdot 3S(1\cdots\ell-2)(\vec{e}_+^{\,\otimes\frac{1}{2}(\ell+m-4)}$$

$$\otimes \vec{e}_-^{\,\otimes\frac{1}{2}(\ell-m-4)} \otimes \vec{e}_3^{\,\otimes 2})] - \cdots -\alpha_i[\tfrac{1}{2}(\ell+m-2i)(\ell-m-2i)S(1\cdots\ell-2)$$

$$(\vec{e}_+^{\otimes\frac{1}{2}(\ell+m-2i-2)} \otimes \vec{e}_-^{\otimes\frac{1}{2}(\ell-m-2i-2)} \otimes \vec{e}_3^{\otimes 2i}) + 2i(2i-1)S(1\cdots -2)$$

$$(\vec{e}_+^{\otimes\frac{1}{2}(\ell+m-2i)} \otimes \vec{e}_-^{\otimes\frac{1}{2}(\ell-m-2i)} \otimes \vec{e}_3^{\otimes(2i-2)})] - \cdots = 0$$

This implies

$$\frac{1}{2}(\ell+m)(\ell-m)-2\alpha_1 = 0$$

$$-\frac{1}{2}\alpha_1(\ell+m-2)(\ell-m-2)-2\cdot 2(2\cdot 2-1)\alpha_2 = 0$$

$$-\frac{1}{2}\alpha_2(\ell+m-4)(\ell-m-4)-2\cdot 3(2\cdot 3-1)\alpha_3 = 0$$

$$\cdots$$

$$-\frac{1}{2}\alpha_{i-1}(\ell+m-2i+2)(\ell-m-2i+2)-2i(2i-1)\alpha_i = 0$$

$$\cdots$$

The expression for α_i follows. The procedure for β_i is obtained similarly. Q.E.D.

Theorem 3: For $m=-\ell,-\ell+1,\cdots,0,\cdots,\ell$; let

$$T_m^\ell \equiv (I_2 \otimes \bar{\Lambda})^{\otimes\ell}\Phi(\ell,m)$$

Then T_m^ℓ are scalar multiples of $T_{m,E}^\ell$ for $\ell+m$ even and of $T_{m,o}^\ell$ for $\ell+m$ odd.

The Projection Operator P_ℓ from $CE_3^{\otimes\ell}$ to $(CE_3^{\otimes\ell})_{s,tr}$

$$\begin{tikzcd}
U_2^{\otimes 2\ell} \arrow[d, "\frac{1}{(2\ell)!}S(1\cdots 2\ell)"] & & CE_3^{\otimes\ell} \arrow[ll, "(-I_2\otimes\Lambda)^{\otimes\ell}"'] \arrow[d, "P_\ell"] \\
L_{2\ell} \arrow[rr, "(I_2\otimes\bar{\Lambda})^{\otimes\ell}"] & & (CE_3^{\otimes\ell})_{s,tr}
\end{tikzcd}$$

From the above diagram, consider the map

$$\frac{1}{(2\ell)!}(I_2 \otimes \bar{\Lambda})^{\otimes \ell} S(1\cdots 2\ell)(-I_2 \otimes \Lambda)^{\otimes \ell}$$

from $CE_3^{\otimes \ell}$ onto $(CE_3^{\otimes \ell})_{s,tr}$ is a projection operator. We now give a derivation of a projection operator P_ℓ by a different method. We believe these two projection operators are identical but have not proved it.

Define the map P_ℓ by

$$P_\ell = N_\ell S(1\cdots\ell)[1+\gamma_1 I_2 C + \gamma_2 I_2^{\otimes 2} C^2 + \cdots + \gamma_i I_2^{\otimes i} C^i + \cdots$$
$$+ \gamma_k I_2^{\otimes k} C^k] S(1\cdots\ell)$$

where $C^i = C(1,2)C(3,4)\cdots C(2i-1,2i)$ and $k = \frac{\ell}{2}$ if ℓ is even, and $k = \frac{\ell-1}{2}$ if ℓ is odd. And N_ℓ, γ_i; $i=1,2,\cdots,k$ are constants.

In order to figure out the constants N_ℓ and γ_i, consider T in $CE_3^{\otimes \ell}$, letting

$$M = S(1\cdots\ell)T$$

implies that M is a symmetric tensor in $CE_3^{\otimes \ell}$. Hence

$$P_\ell T = N_\ell S(1\cdots\ell)[1+\gamma_1 I_2 C+\gamma_2 I_2^{\otimes 2} C^2 + \cdots + \gamma_i I_2^{\otimes i} C^i + \cdots$$
$$+\gamma_k I_2^{\otimes k} C^k] S(1\cdots\ell)T$$
$$= N_\ell\{S(1\cdots\ell)M+\gamma_1 S(1\cdots\ell)I_2 \otimes (CM)+\gamma_2 S(1\cdots\ell)I_2^{\otimes 2} \otimes (C^2 M) + \cdots$$
$$+ \gamma_i S(1\cdots\ell)I_2^{\otimes i} \otimes (C^i M) + \cdots + \gamma_k S(1\cdots\ell)I_2^{\otimes k} \otimes (C^k M)\}$$

Consider the contraction $C[S(1\cdots\ell)I_2^{\otimes i} \otimes (C^i M)]$. The number of ways that one of the tensors I_2 can be selected and placed in the first two positions is 2i. Since $CI_2=3$, each way gives a term $3\mathcal{S}(1\cdots\ell-2)I_2^{\otimes(i-1)}\otimes(C^iM)$. The number of ways that a file from two different tensors I_2 can be selected and placed in the first two positions is $4i(i-1)$. A single contraction involving one file of I_2 causes the I_2 to disappear as in the example: $C(1,2)\vec{a}I_2 = \vec{a}$. It follows that each way gives a term $\mathcal{S}(1\cdots\ell-2)I_2^{\otimes(i-1)}\otimes(C^iM)$. The number of ways that a file from one of the tensors I_2 and a file from C^iM can be selected and placed in the first two positions is $4i(\ell-2i)$. It follows that each way gives a term $\mathcal{S}(1\cdots\ell-2)I_2^{\otimes(i-1)}\otimes(C^iM)$. Adding all the above contributions gives $2i(2\ell+1-2i)\mathcal{S}(1\cdots\ell-2)I_2^{\otimes(i-1)}\otimes(C^iM)$. The number of ways that two files from C^iM can be selected and placed in the first two positions is $(\ell-2i)(\ell-2i-1)$. This results in $(\ell-2i)(\ell-2i-1)$ $\mathcal{S}(1\cdots\ell-2)I_2^{\otimes i}\otimes(C^{i+1}M)$. Thus we have

$$C[\mathcal{S}(1\cdots\ell)I_2^{\otimes i}\otimes(C^iM)] = 2i(2\ell+1-2i)\mathcal{S}(1\cdots\ell-2)I_2^{\otimes(i-1)}\otimes(C^iM)$$

$$+ (\ell-2i)(\ell-2i-1)\mathcal{S}(1\cdots\ell-2)I_2^{\otimes i}\otimes(C^{i+1}M)$$

then

$$C(P_\ell T) = N_\ell\{\ell(\ell-1)\mathcal{S}(1\cdots\ell-2)(CM)+\gamma_1[2\cdot1\cdot(2\ell+1-2\cdot1)\mathcal{S}(1\cdots\ell-2)$$

$$(CM)+(\ell-2\cdot1)(\ell-2\cdot1-1)\mathcal{S}(1\cdots\ell-2)I_2\otimes(C^2M)]$$

$$+\gamma_2[2\cdot2\cdot(2\ell+1-2\cdot2)\mathcal{S}(1\cdots\ell-2)I_2\otimes(C^2M)+(\ell-2\cdot2)(\ell-2\cdot2-1)\mathcal{S}(1\cdots\ell-2)$$

$$I_2^{\otimes2}\otimes(C^3M)] + \cdots +\gamma_i[2i(2\ell+1-2i)\mathcal{S}(1\cdots\ell-2)I_2^{\otimes(i-1)}\otimes(C^iM)$$

$$+ (\ell-2i)(\ell-2i-1)\mathcal{S}(1\cdots\ell-2)I_2^{\otimes i}\otimes(C^{i+1}M) + \cdots$$

$$+\gamma_k[2k(2\ell+1-2k)\mathcal{S}(1\cdots\ell-2)I_2^{\otimes(k-1)}\otimes(C^kM)]$$

Setting $C(P_\ell T) = 0$ gives

$$(\ell-1) + \gamma_1[2(2\ell-1)] = 0$$

$$\gamma_1[2(2\ell-1)] + \gamma_2[4(2\ell-3)] = 0$$

$$\cdot\ \cdot\ \cdot$$

$$\gamma_{i-1}[(\ell+2-2i)(\ell+1-2i)] + \gamma_i[2i(2\ell+1-2i)] = 0$$

$$\cdot\ \cdot\ \cdot$$

therefore

$$\gamma_1 = -\frac{\ell(\ell-1)}{2(2\ell-1)}$$

$$\gamma_2 = -\frac{(\ell-2)(\ell-3)}{4(2\ell-3)}$$

$$\cdot\ \cdot\ \cdot$$

$$\gamma_i = -\frac{(\ell+2-2i)(\ell+1-2i)}{2i(2\ell+1-2i)}$$

for $i=1,2,\cdots,k$.

To find N_ℓ we consider

$$P_\ell P_\ell T = N_\ell S(1\cdots\ell)(1+\gamma_1 I_2 C+\gamma_2 I_2^{\otimes 2}C^2+ \cdots +\gamma_i I_2^{\otimes i}C^i + \cdots$$
$$+\gamma_k I_2^{\otimes k}C^k)S(1\cdots\ell)P_\ell T$$
$$= N_\ell(\ell!)S(1\cdots\ell)P_\ell T$$

since all terms drop out except the first due to the fact that $CP\ T=0$. Then

$$P_\ell P_\ell T = N_\ell(\ell!)^2 S(1\cdots\ell)\ N_\ell S(1\cdots\ell)(1+\gamma_1 I_2 C+\gamma_2 I_2^{\otimes 2}C^2 + \cdots$$
$$+\gamma_i I_2^{\otimes i}C^i + \cdots +\gamma_k I_2^{\otimes k}C^k)S(1\cdots\ell)T$$
$$= N_\ell(\ell!)^2 P_\ell T$$

The requirement that $P_\ell P_\ell T = P_\ell T$ implies

$$N_\ell = (\ell!)^2$$

From the above derivation, we know that the map p_ℓ sends an arbitrary tensor T in $CE_3^{\otimes \ell}$ into $P_\ell T$, a symmetric traceless tensor which is an element of $(CE_3^{\otimes \ell})_{s,tr}$. Thus P_ℓ in fact is a projection operator of $CE_3^{\otimes \ell}$ onto $(CE_3^{\otimes \ell})_{s,tr}$. We state the following theorem which we have not proved:

Theorem 1:

$$P_\ell = \frac{1}{(2\ell)!} (I_2 \otimes \overline{\Lambda})^{\otimes \ell} S(1 \cdots 2\ell)(-I_2 \otimes \Lambda)^{\otimes \ell}$$

Map of Tensor Spaces Onto Function Spaces

We have previously described a map $(I_2 \otimes \overline{\Lambda})^{\otimes \ell}$ which takes $L_{2\ell} \equiv (U_2^{\otimes 2\ell})_s$ one-to-one onto $(CE_3^{\otimes \ell})_{s,tr}$. The orthonormal basis of $L_{2\ell}$ consisting of simultaneous eigentensors $\Phi(\ell,m)$ of $D(J_z)$ and $\Delta(J^2)$ are mapped onto an orthonormal basis of $(CE_3^{\otimes \ell})_{s,tr}$ consisting of simultaneous eigentensors $T_\ell^m = (I_2 \otimes \overline{\Lambda})^{\otimes \ell}\Phi(\ell,m)$ of $D(J_z)$ and $\Delta(J^2)$. We now describe a map

$$(CE_3^{\otimes \ell})_{s,tr} \rightarrow P_\ell(E_3)$$

which is one-to-one onto, where $P_\ell(E_3)$ is the set of complex polynomials on E_3 which are homogeneous of degree ℓ and harmonic. This map is as follows:

$$T \varepsilon (CE_3^{\otimes \ell})_{s,tr} \rightarrow t(\vec{r}) = T \bullet \vec{r}^{\otimes \ell} \varepsilon P_\ell(E_3) .$$

Let

$$t_m^\ell(\vec{r}) = T_m^\ell \bullet \vec{r}^{\otimes \ell} .$$

Thus under this map, the simultaneous eigentensors T_m^ℓ of $D(J_z)$ and $\Delta(J^2)$ go into $t_m^\ell(\vec{r})$ which are simultaneous eigenfunctions of the differential operators L_z, L^2 which represent J_z and J^2 on the spaces of functions on

E_3 and are given as follows

$$L_x = -i\vec{e}_1 \cdot \vec{r} \times \vec{\nabla},\ L_y = -i\vec{e}_2 \cdot \vec{r} \times \vec{\nabla},\ L_z = -i\vec{e}_3 \cdot \vec{r} \times \vec{\nabla}$$

and

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

We believe but have not proved that

$$t_m^\ell(\vec{r}) = (\ell!)\sqrt{\frac{8\pi}{(2\ell+1)!}}\ r^\ell Y_\ell^m(\theta,\phi)$$

where $Y_\ell^m\ (\theta,\phi)$ are the spherical harmonics which we define soon. The reason for this belief is that Y_ℓ^m are simultaneous eigenfunctions of L_z, L^2. The constant may be off by factors of the form $e^{i\lambda}$ for λ real.

Now for $\vec{r}$ in E_3, expressing $\vec{r}$ in terms of spherical coordinates, we have

$$\vec{r} = r(\sin\theta\ \cos\phi\ \vec{e}_1 + \sin\theta\ \sin\phi\ \vec{e}_2 + \cos\theta\ \vec{e}_3)$$

The Legendre polynomials are given by the Rodrigues formula as

$$\mathcal{P}_\ell(z) = \frac{1}{2^\ell(\ell!)}\frac{d^\ell}{dz^\ell}(z^2-1)^\ell\ ;\ (\ell = 0,1,2,\cdots)$$

and the associated Legendre functions are

$$\mathcal{P}_\ell^m(z) = (1-z^2)^{\frac{m}{2}}\ \frac{d^m}{dz^m}\mathcal{P}_\ell(z)$$

$$= \frac{1}{2^\ell(\ell!)}(1-z^2)^{\frac{m}{2}}\ \frac{d^{\ell+m}}{dz^{\ell+m}}(z^2-1)^\ell$$

for $m=-\ell,-\ell+1,\cdots,0,\cdots,\ell-1,\ell$. The spherical harmonics are

$$Y_\ell^m(\theta,\phi) = (-1)^m\left[\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}\right]^{\frac{1}{2}} P_\ell^m(\cos\theta)e^{im\phi}$$

Inner Products Under the Map into Function Space

Consider a symmetric tensor S in $(CE_3^{\otimes \ell})_s$.

Theorem 1: If $S \odot \vec{a}^{\otimes \ell} = 0$ for arbitrary $\vec{a} \in E_3$ then $S = 0$.

Proof: Suppose $S \odot \vec{a}^{\otimes \ell} = 0$ for arbitrary $\vec{a} \in E_3$, then

$$S \odot \Big(\sum_{k=1}^{\ell} \alpha_k \vec{c}_k\Big)^{\otimes \ell} = 0$$

for arbitrary $\alpha_1, \alpha_2, \cdots, \alpha_\ell$ in C and $\vec{c}_1, \vec{c}_2, \cdots, \vec{c}_\ell$ in E_3. Hence

$$\frac{\partial^\ell}{\partial\alpha_1 \partial\alpha_2 \cdots \partial\alpha_\ell}\Big[S \odot \Big(\sum_{k=1}^{\ell} \alpha_k \vec{c}_k\Big)^{\otimes \ell}\Big] = 0$$

which gives

$$S \odot [\mathcal{S}(1\cdots\ell)(\vec{c}_1 \otimes \vec{c}_2 \otimes \cdots \otimes \vec{c}_\ell)]$$

$$= [\mathcal{S}(1\cdots\ell)S] \odot (\vec{c}_1 \otimes \vec{c}_2 \otimes \cdots \otimes \vec{c}_\ell)$$

$$= (\ell!)\ S \odot (\vec{c}_1 \otimes \vec{c}_2 \otimes \cdots \otimes \vec{c}_\ell) = 0$$

Since $\vec{c}_1, \vec{c}_2, \cdots, \vec{c}_\ell$ are arbitrary, it follows that S=0. Q.E.D.

Corollary 1: For S,T in $(CE_3^{\otimes \ell})_s$, S=T if and only if $S \odot \vec{a}^{\otimes \ell} = T \odot \vec{a}^{\otimes \ell}$ for all $\vec{a} \in E_3$.

Theorem 2:

$$\int_0^\pi \int_0^{2\pi} \vec{r}^{\otimes 2\ell}\, d\Omega = \frac{4\pi}{(2\ell+1)!}\, r^{2\ell} \mathcal{S}(1\cdots 2\ell)\, I_2^{\otimes \ell}$$

where $d\Omega = \sin\theta\, d\phi d\theta$.

Proof: For arbitrary $\vec{a} \in E_3$, we obtain

$$\left(\int_0^{\pi}\int_0^{2\pi} \vec{r}^{\,\otimes 2\ell} d\Omega\right) \boldsymbol{\circ}\, \vec{a}^{\,\otimes 2\ell}$$

$$= \int_0^{\pi}\int_0^{2\pi} (\vec{r}^{\,\otimes 2\ell} \boldsymbol{\circ}\, \vec{a}^{\,\otimes 2\ell}) d\Omega$$

$$= (ra)^{2\ell} \int_0^{\pi}\int_0^{2\pi} \cos^{2\ell}\theta' \ \sin\theta \ d\phi d\theta$$

$$= \frac{4\pi}{2\ell+1} (ra)^{2\ell}$$

since the integral is a linear operator. We let $\vec{a}$ be the z-axis of the coordinate system. Also

$$\left[\frac{4\pi}{(2\ell+1)!} r^{2\ell} \mathcal{S}(1\cdots 2\ell) I_2^{\otimes \ell}\right] \boldsymbol{\circ}\, \vec{a}^{\,\otimes 2\ell}$$

$$= \frac{4\pi}{(2\ell+1)!} r^{2\ell} (I_2^{\otimes \ell} \boldsymbol{\circ}\, \mathcal{S}(1\cdots 2\ell) \vec{a}^{\,\otimes 2\ell}$$

$$= \frac{4\pi}{2\ell+1} r^{2\ell} \ (I_2^{\otimes \ell} \boldsymbol{\circ}\, \vec{a}^{\,\otimes 2\ell})$$

$$= \frac{4\pi}{2\ell+1} r^{2\ell} \ (\vec{a} \cdot \vec{a})^{\ell}$$

$$= \frac{4\pi}{2\ell+1} (ra)^{2\ell}$$

By the above Corollary 1, the theorem follows. Q.E.D.

Let S be in $(CE_3^{\otimes 2\ell})_s$, and define $s(\theta,\phi) = r^{-2\ell} S \boldsymbol{\circ}\, \vec{r}^{\,\otimes 2\ell}$.

Theorem 3:

$$\int_0^{\pi}\int_0^{2\pi} \sin\theta \ s(\theta,\phi) d\phi d\theta = \frac{4\pi}{2\ell+1} C^{\ell}(S)$$

Proof:

$$S \boldsymbol{\circ}\, [\mathcal{S}(1\cdots 2\ell) I_2^{\otimes \ell}]$$

$$= [\mathcal{S}(1\cdots 2\ell) S] \boldsymbol{\circ}\, I_2^{\otimes \ell}$$

$$= (2\ell)!\ S \odot I_2^{\otimes \ell}$$

$$= (2\ell)!\,C^{\ell}(S)$$

since

$$(\vec{a}_1 \otimes \vec{a}_2 \otimes \vec{a}_3 \otimes \vec{a}_4 \otimes \cdots \otimes \vec{a}_{2\ell-1} \otimes \vec{a}_{2\ell}) \odot I_2^{\otimes \ell}$$

$$= [(\vec{a}_1 \otimes \vec{a}_2) \odot I_2][(\vec{a}_3 \otimes \vec{a}_4) \odot I_2] \cdots [(\vec{a}_{2\ell-1} \otimes \vec{a}_{2\ell}) \odot I_2]$$

$$= C(\vec{a}_1 \otimes \vec{a}_2) C(\vec{a}_3 \otimes \vec{a}_4) \cdots\ C(\vec{a}_{2\ell-1} \otimes \vec{a}_{2\ell})$$

$$= C^{\ell}(\vec{a}_1 \otimes \vec{a}_2 \otimes \vec{a}_3 \otimes \vec{a}_4 \otimes \cdots \otimes \vec{a}_{2\ell-1} \otimes \vec{a}_{2\ell})$$

thus

$$\int_0^{\pi} \int_0^{2\pi} \sin\theta\ s(\theta,\phi)\,d\phi\,d\theta$$

$$= \int_0^{\pi} \int_0^{2\pi} \sin\theta\ (r^{-2\ell} S \odot \vec{r}^{\otimes 2\ell})\,d\phi\,d\theta$$

$$= S \odot \Big[\int_0^{\pi} \int_0^{2\pi} \vec{r}^{-2\ell} \sin\theta\ \vec{r}^{\otimes 2\ell}\,d\phi\,d\theta\Big]$$

$$= S \odot \Big[\int_0^{\pi} \int_0^{2\pi} \vec{r}^{-2\ell} \vec{r}^{\otimes 2\ell}\ d\Omega\Big]$$

$$= S \odot \Big[\frac{4\pi}{(2\ell+1)!}\ \mathcal{S}(1\cdots 2\ell)\ I_2^{\otimes \ell}\Big]$$

by Theorem 2 above. Hence

$$\int_0^{\pi} \int_0^{2\pi} \sin\theta\ s(\theta,\phi)\,d\phi\,d\theta$$

$$= \frac{4\pi}{(2\ell+1)!}\ S \odot [\mathcal{S}(1\cdots 2\ell) I_2^{\otimes \ell}]$$

$$= \frac{4\pi}{2\ell+1}\ C^{\ell}(S)$$

This proves Theorem 3. Q.E.D.

Theorem 4: If S and T are in $(CE_3^{\otimes \ell})_{s,tr}$ and

$$s(\theta,\phi) = r^{-\ell} S \bigcirc \vec{r}^{\otimes \ell}$$

$$t(\theta,\phi) = r^{-\ell} T \bigcirc \vec{r}^{\otimes \ell}$$

then

$$S \bigcirc T = \frac{(2\ell+1)!}{8\pi(\ell!)^2} \int_0^{\pi} \int_0^{2\pi} \sin\theta \; s(\theta,\phi) t(\theta,\phi) d\phi d\theta$$

Proof: First we claim that

$$C^{\ell}[\mathcal{S}(1\cdots 2\ell)(S \otimes T)] = 2(\ell!)^2 \, S \bigcirc T$$

To prove this, letting $S = \mathcal{S}(1\cdots\ell)(\vec{a}_1 \otimes \vec{a}_2 \otimes \cdots \otimes \vec{a}_\ell)$ and $T = \mathcal{S}(1\cdots\ell)(\vec{a}_{\ell+1} \otimes \vec{a}_{\ell+2} \otimes \cdots \otimes \vec{a}_{2\ell})$, then

$$C^{\ell}[\mathcal{S}(1\cdots 2\ell)(S \otimes T)]$$

$$= C^{\ell}\{\mathcal{S}(1\cdots 2\ell)[\mathcal{S}(1\cdots\ell)(\vec{a}_1 \otimes \vec{a}_2 \otimes \cdots \otimes \vec{a}_\ell) \otimes \mathcal{S}(1\cdots\ell)$$

$$(\vec{a}_{\ell+1} \otimes \vec{a}_{\ell+2} \otimes \cdots \otimes \vec{a}_{2\ell})]\}$$

$$= C^{\ell}\{\mathcal{S}(1\cdots 2\ell)[\mathcal{S}(1\cdots\ell)\mathcal{S}(\ell+1\cdots 2\ell)(\vec{a}_1 \otimes \vec{a}_2 \otimes \cdots \otimes \vec{a}_\ell \otimes \vec{a}_{\ell+1}$$

$$\otimes \cdots \otimes a_{2\ell})]\}$$

$$= (\ell!)^2 C^{\ell}\{\mathcal{S}(1\cdots 2\ell)(\vec{a}_1 \otimes \vec{a}_2 \otimes \cdots \otimes \vec{a}_{2\ell})\}$$

$$= 2(\ell!)^2 [\mathcal{S}(1\cdots\ell)(\vec{a}_1 \otimes \vec{a}_2 \otimes \cdots \otimes \vec{a}_\ell)] \bigcirc [\mathcal{S}(1\cdots\ell)(\vec{a}_{\ell+1} \otimes \cdots \otimes \vec{a}_{2\ell})]$$

$$= 2(\ell!)^2 \, S \bigcirc T$$

since S,T are both traceless, so the only nonvanishing terms are those pairs $\vec{a}_i \cdot \vec{a}_j$ with $1 \le i \le \ell$ and $\ell+1 \le j \le 2\ell$.

To prove the main theorem. let $W = \mathcal{S}(1\cdots 2\ell)(S \otimes T)$ and $w(\theta,\phi) = r^{-2\ell} W \bigcirc \vec{r}^{\otimes 2\ell}$, then from Theorem 3, we obtain

$$\int_0^\pi \int_0^{2\pi} \sin\theta \; w(\theta,\phi)\,d\phi\,d\theta$$

$$= \frac{4\pi}{2\ell+1} \, C^\ell(W)$$

$$= \frac{8\pi}{2\ell+1} \, (\ell!)^2 (S \bigcirc T)$$

thus

$$\int_0^\pi \int_0^{2\pi} \sin\theta \; w(\theta,\phi)\,d\phi\,d\theta$$

$$= \int_0^\pi \int_0^{2\pi} \sin\theta \; r^{-2\ell}(W \bigcirc \vec{r}^{\otimes 2\ell})\,d\phi\,d\theta$$

$$= \int_0^\pi \int_0^{2\pi} \sin\theta \; r^{-2\ell}[\mathcal{S}(1\cdots 2\ell)(S \otimes T)] \bigcirc \vec{r}^{\otimes 2\ell} \; d\phi\,d\theta$$

$$= \int_0^\pi \int_0^{2\pi} \sin\theta \; (2\ell)!(r^{-\ell}S \bigcirc \vec{r}^{\otimes \ell})(r^{-\ell}T \bigcirc \vec{r}^{\otimes \ell})\,d\phi\,d\theta$$

$$= (2\ell)! \int_0^\pi \int_0^{2\pi} \sin\theta \; s(\theta,\phi)t(\theta,\phi)\,d\phi\,d\theta$$

implies that

$$\int_0^\pi \int_0^{2\pi} \sin\theta \; s(\theta,\phi)t(\theta,\phi)\,d\phi\,d\theta$$

$$= \frac{8\pi(\ell!)^2}{(2\ell+1)!} \, (S \bigcirc T)$$

This proves Theorem 4. Q.E.D.

Another result which follows from this is

$$\int_0^\pi \int_0^{2\pi} \sin\theta \; \overline{s(\theta,\phi)} \; t(\theta,\phi)\,d\phi\,d\theta = \frac{8\pi(\ell!)^2}{(2\ell+1)!} <S|T>$$

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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